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A New Approach to Interacting Fields

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Abstract

A model for a description of interaction, which involves particle creation, can be given as follows:

(1) A smooth finite-dimensional manifold M constitutes the configuration space of some interacting system.
(2) The concept of an interacting field is formulated in terms of two-component objects which consist of a physical and a topological field component which are 'derived' from M.
(3) Interaction is described in terms of the topological linking number of the topological field components and in terms of the intrinsic field equations.

This scheme provides a geometrical description of strong interactions and gives a structural analysis of Gell-Mann current fields. A differential topological formulation of Noether's Theorem can be obtained. Moreover a consistent description of electromagnetic interactions which sheds a new light on the mechanism of virtual processes is available. This description results in an estimate of the fine structure constant $\alpha = e^2/(\hbar \cdot c)$.

1. *Introduction*

The mathematical difficulties connected with the quantisation of non-linear field theories are presumably due to a formalism which is trying to describe non-linear systems, i.e. interacting fields, in terms of linear operators (asymptotic free fields, 'bare' fields etc.) and linear configuration spaces. Other key problems arise in connection with interaction Hamiltonians such as $H_I = ej_\mu A^\mu$, $j_\mu = \bar{\psi}\gamma_\mu\psi$ (A^μ and ψ stand for the quantised photon and electron field respectively), which represent the most basic interactions of field theory, i.e. the current-type interactions. It is known, however, that the expression for H_I fails to make mathematical sense because of the lack of meaning for trilinear products of operator-valued fields.

To arrive at a way of treating interacting systems in an intrinsic fashion, it seems reasonable to proceed within some suitable geometric framework. It is our aim, therefore, to provide a formulation of the concept of an interacting field in terms of pure geometry which consists of

(I)

(1) An arbitrary finite-dimensional smooth configuration manifold M whose differential geometric structure accounts for the interaction in conjunction with
(2) fields, which are defined as pairings

$$(\omega^p, c_p) \qquad \text{(v. Westenholz, 1972)} \tag{1}$$

($\omega^p \in F^p(M)$, the space of p-forms on M and $c_p \in C_p(M)$, the space of p-chains on M and
(3) a description of interacting fields of type (1), $(\omega_1{}^1, c_1{}^1)$, $(\omega_2{}^1, c_2{}^1)$, ... $(\omega_n{}^1, c_n{}^1)$, $\omega_i{}^1 \in F^1(M)$, $c_1{}^j \in C_1(M)$, in terms of the topological linking number $l(c_1{}^i, c_1{}^j)$. The field equations which correspond to this interaction are obtained in an intrinsic form by means of the coordinate-free operators d (exterior differentiation), $i_X = \lrcorner$ (contraction by a vector field X) and L_X (Lie derivative), such that only the 'physical' field component ω of (1) is involved.

With a certain amount of over-simplification one may say that the present model provides an approach to an elementary particle description where particle physics is reduced to topological issues. A first phenomenological programme along these lines has been achieved by Jehle (1971). Our model departs, however, in important ways from his treatment.

Remark 1. With an approach to field theory in terms of smooth manifolds it is impossible to describe a field by giving its components with respect to a single set of coordinates. An intrinsic description of physical laws is provided, however, in terms of differential forms and the corresponding dual objects, the vector fields. The case of skew-symmetric covariant tensor fields, which are the most frequently encountered in physics, accounts for this. Electromagnetic theory, the Hamilton–Jacobi theory, the Yang–Mills theory, etc., can be given a neat and concise formulation in terms of these intrinsic objects.

Remark 2. The concept of a topological field (1) is illustrated by a conservative force field (ω^1, c_1), $\omega^1 = \Sigma_i F_i \, dx^i$, where $\int_{c_1} \omega^1 = 0 \; \forall c_1$, c_1: closed curve, yields $\exists \varphi: \omega = -d\varphi$, $\varphi \in F^0(M)$. That is, the force field (ω^1, c_1) may be regarded as being associated with some geometry M, subject to the constraints

$$\Pi_1(M) = H_1(M) = H^1(M) = 0 \tag{2}$$

($\Pi_1(M)$ denotes the Poincaré group, $H_1(M)$ and $H^1(M)$ the first homology and cohomology groups of M respectively). More generally, fields of the type (1) may be regarded as being 'derived' from some suitable geometry. This amounts to saying that the study of the homotopy groups $\Pi_p(M)$, $p = 1, 2, \ldots$, the homology and cohomology groups $H_p(M)$ and $H^p(M)$ of M exhibit which way the properties of the geometry imply the properties of some field of type (1).

This issue of potential function can also be discussed in terms of vector fields and homology. Suppose M is equipped with a Riemannian metric $\langle\ ,\ \rangle$ and

$$X \in \mathfrak{X}(M) \tag{3}$$

is a vector field. If (3) corresponds to the closed one form ω_X, given by

$$\langle X, Y\rangle = \omega_X(Y)\ \forall Y \in T_p(M)\ \forall p \in M \tag{4}$$

then $\omega_X(Y)$ is called the work of the field of force Y. Either of two cases may occur: (1) If ω_X is homologous to zero in the one-dimensional cohomology group $H^1(M, \mathbb{R})$, then $\omega = -d\varphi$, i.e.

$$X = -\operatorname{grad}\varphi; \qquad \langle \operatorname{grad}\varphi, Y\rangle = d\varphi(Y) \tag{3}$$

or (2) if ω_X is not homologous to zero in $H^1(M, \mathbb{R})$ then the potential is 'multi-valued', i.e. it is defined up to multiples of periods.

The aim of this paper is to study certain classes of fields of the type (1) and to investigate what new insight into physics can be gained. In Section 2, we show that a Cartan-Euler field of type (1) essentially characterises the whole dynamics of some mechanical system. A differential topological version of Noether's theorem, which extends to systems with infinite degree of freedom, is thus obtained. Section 3 is devoted to the study of Aharanow-Bohm fields which are related to the Aharanow-Bohm effect (v. Westenholz, 1973; Aharonow & Bohm, 1959). In studying Aharanow-Bohm fields one is led to the concept of path-dependent matter field variables (Mandelstam, 1962), which motivate vividly an approach to Yang-Mills fields (which are generalised Aharanow-Bohm fields) in terms of some curved geometry. As the differential geometric structure involved will be some fibre bundle, it turns out, that, on a rigorous level, strong interactions can be described with recourse to a formal correspondence principle between fibre bundles and the Yang-Mills theory. This issue will be developed in Section 4. Our Section 5 is devoted to a differential topological version of Noether's theorem for systems with infinite degrees of freedom. This leads to a structural interpretation of currents within the framework of fibre bundles and introduces Gell-Mann current fields which are of the type (1). In our final section, Section 6, interacting quarks, represented by the linkage of quantised loops, which are the topological Yang-Mills field components, are analysed. Particularly for electromagnetic interactions, i.e. for Aharanow-Bohm fields, the magnitude of the interaction constant $\alpha = (e^2/\hbar c)$ is obtained.

2. *The Cartan-Euler Field*

A topological Cartan-Euler field can be defined by

$$(\theta, c_1) \tag{5}$$

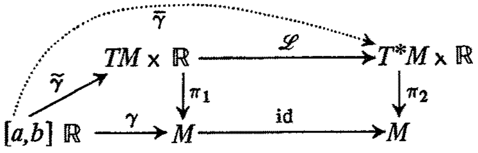
where

$$\theta = p_i dq^i - H.\, dt \in F^1(T^*M \times \mathbb{R}) \tag{6}$$

denotes the Cartan 1-form and $c_1 \in C_1(T^*M \times \mathbb{R})$ the one-chain which corresponds to the Euler extremal of the corresponding variational problem on the evolution space $P = T^*M \times \mathbb{R}$ (T^*M denotes the cotangent bundle). It turns out, that the field (5) accounts for the whole dynamics of a system by virtue of

Hamilton's variational principle, which states that the integral of action $S = \int_{c_1} p_i dq^i - H.dt$ (which corresponds to the usual form $\int (T - V) \cdot dt$ in which this principle is quoted) has a stationary value for the natural motion when compared with adjacent motions having the same end events.

The corresponding canonical variational problem is defined by the manifold $P = T^*M \times \mathbb{R}$, the Cartan 1-form (6), a differential ideal $I \subset F(M)$ and an integral manifold (ψ, N) (Hermann, 1968). This leads, in terms of the adjacent diagram,



$\gamma = [a, b] \to M \gamma \in C^1.$
$\bar{\gamma} = \mathscr{L} \circ \gamma, \mathscr{L}$ = Legendre transform

$$\mathscr{L}(q^1 \ldots q^n, \dot{q}_1 \ldots \dot{q}_n, t) = \left(q^1 \ldots q^n, \frac{\partial L}{\partial q_1} \ldots \frac{\partial L}{\partial q_n}, t\right).$$

L is a regular Lagrangian.

Figure 1.

to the study of the behaviour of the functional

$$\gamma \to I(\gamma) = \int_{[a,b]} L \circ \tilde{\gamma}\, dt = \int_{[a,b]} \bar{\gamma}^* \theta = \int_{\bar{\gamma}_*[a,b]} \theta \tag{7}$$

which must be stationary.

Formula (7) is the abridged notation for

$$(\theta, \bar{\gamma}_*[a, b]) \to \int_{\bar{\gamma}_*[a,b]} \theta \tag{8}$$

The first variation formula yields the relationship

$$X \lrcorner\, d\theta(L)|_{c_1} = 0 \qquad (\lrcorner \text{ denotes the contraction of } d\theta \text{ by the vector field } X) \tag{9}$$

That is, there is a unique vector field X satisfying (9) on $T^*M \times \mathbb{R}$, called the Euler vector field. The corresponding differential equations of motion are the Euler–Lagrange equations. c_1 is the extremal of $d\theta(L)$ (or equivalently of the Lagrangian L) iff (9) holds. In summary:

A topological model for interacting mechanical systems is given in terms of Scheme (I) of Section 1 by:

(1) The differential geometric structure of the evolution space $P = \{(p, q, t)\} = T^*M \times \mathbb{R}$ in conjunction with

(2) the Gartan-Euler field (θ, c_1) where

$$\theta = p_i dq^i - H.dt$$

$$c_1 = \bar{\gamma}_*[a, b]$$

and

(3) the Euler–Lagrange differential equations of motion $X \lrcorner\, d\theta(L)|_{c_1} = 0$.

This model fully accounts for the dynamics of a system under the influence of some force field $\omega = F_i . dq^i = -dV$.

Remark 3. On account of $V = V(q)$, which represents the influence of a potential, the interaction is already characterised without having recourse to the linking number.

Remark 4. As shown by Gallissot (1951), Newton's equations of motion can be obtained from some Galilean-invariant exterior form, which in fact is just the exterior derivative of the Cartan form (6). Indeed, let $P = T^*M \times \mathbb{R} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ be the evolution space for a particle and

$$X = \sum \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}$$

then one has $i_X\omega = 0$ for $\omega = dp_i dq^i - dH \cdot dt$. Moreover

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Proposition 1 (*Gallissot*). Newton's law of motion for a particle m under the influence of a force field $\vec{F} = (F_i)$ is given by

$$i_X\omega = X \rfloor \omega \equiv 0 \qquad \text{where} \qquad X = \sum \frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i} + \frac{\partial}{\partial t} \tag{10}$$

$$\omega = \sum \delta_{ij}(m\,.dv^i - F^i dt) \wedge (dx^j - v^j\,.dt)$$

$$\text{(Galilean-invariant 2-form)} \tag{11}$$

Proof. By virtue of the definition $i_X\omega \equiv X^\alpha[\partial\omega/\partial(dx^\alpha)]$ we obtain:

$$i_X\omega \equiv 0 \Rightarrow m\,.dv^i - F^i\,.dt = (dx^i - v^i\,.dt) = 0$$

Expression (11) is related to Cartan's form (6), since (11) yields:

$$\omega = m\delta_{ij}\, dv^i \wedge dx^j - m\delta_{ij}\, v^i dv^j \wedge dt + \delta_{ij} F_i dx^j \wedge dt$$

If $d\omega = 0$, then $\omega^1 = \delta_{ij} F_i dx^j = \Sigma_i F_i dx^i$ must be closed.

Since the first Betti number

$$\beta_1(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}) = 0 \Rightarrow \exists V: \qquad \omega^1 = \sum F_i dx^i = -dV$$

$$\begin{aligned}(11) \Rightarrow \omega &= m\delta_{ij} dv^i \wedge dx^j + \delta_{ij}[F_i dx^j - mv^j dv^i] \wedge dt \\ &= m\, dv^i \wedge dx^i + [-dV - mv^j dv^j] dt \\ &= m\, dv^i \wedge dx^i - dH \wedge dt \\ &= dp_i\,.dq^i - dH \wedge dt = d\theta(L),\end{aligned}$$

since $H = \frac{1}{2}mv^{i2} + V$, $dH = \Sigma\, mv^i dv^i + dV$.

It has been emphasised in our introductory section that the differential geometrical set-up is also well suited to dealing with Noether's theorem. We are going to analyse this point now.

Let a state of a system be defined to be a point of the cotangent bundle $T^*(M)$ in the sense that giving such a state at one time determines the future time evolution of the system. More precisely: a state $(q_1(t), \ldots, q_n(t), p_1(t) \ldots p_n(t))$ at a time $t > 0$ is uniquely determined by

> (a) a physical law $U_t \in \{U_t | -\infty < t < +\infty\}$ (the dynamical one-parameter group of the system), and
> (b) the state $(q_1(0) \ldots q_n(0) \ldots p_n(0))$ at $t = 0$.

That is, $(q_1(t) \ldots p_n(t)) = U_t(q_1(0) \ldots p_n(0))$ denotes the state at $t > 0$ and each $(q(t), p(t))$ lies on one and only one U_t-orbit

$$0(p, q) = \{U_t(p, q) | (p, q) \in T^*M \text{ fixed}\} \quad t \in (-\infty, +\infty)$$

This means that the points $(p, q) \in T^*M$ are in one–one correspondence with the Cauchy data for the differential equations determining the time evolution of the system. Alternately, a state of a system may be considered as a curve in $P = T^*M \times \mathbb{R}$, that is, an integral curve of the vector field

$$X = \sum_i \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}$$

That is, we have

> *Definition* 1. A state in $(T^*M \times \mathbb{R}, d\theta(L))$ is any maximal submanifold $c \subset T^*M \times \mathbb{R}$ such that $X \,\lrcorner\, d\theta(L)|_c = 0 \;\forall X$.

In this context one can deal with problems admitting given groups of symmetries. Let P be the phase space and G any Lie group. Diff (P) denotes the group of all C^∞-diffeomorphisms of P. Symmetry groups of some systems shall be specified as follows:

> *Definition* 2. An action of G on P is a group homomorphism $G \xrightarrow{\varphi} \text{Diff}(P)$ such that the mapping
>
> $$\Phi : G \times P \to P : (g, p) \to \varphi(g)(p) \text{ is } C^\infty \tag{12}$$

Condition (12) defines a dynamical symmetry group of some dynamical system characterised by a Lagrangian L iff G preserves the form $\omega = d\theta(L)$. Otherwise stated

$$G \text{ is called a group of symmetries} \Leftrightarrow \varphi^*(g)\omega \,\forall g = \omega$$

i.e. G leaves ω invariant. That is

$$\text{Diff}(P, \omega) := \{\varphi \in \text{Diff}(P) | \varphi^*\omega = \omega\} \tag{13}$$

is the generalised symplectic group.

The infinitesimal counterpart to Definition 2 is

> *Definition* 3. A vector field on P (i.e. $Y \in \mathfrak{X}(P)$) generates a symmetry of L (i.e. is called an infinitesimal symmetry of L) if
>
> $$Y(d\theta(L)) = 0 \tag{14}$$

Constants of motion are now obtained by

Proposition 2 (*Noether*). Let $Y \in d\varphi_g(\mathfrak{g}) \subset \mathfrak{X}(P)$ be a symmetry field of L. Then there exists an observable f which is a constant of motion, i.e. which is constant along the characteristic curves of $d\theta(L)$.

Proof. Suppose $\beta_1(P) = 0$,

$$Y(d\theta) = Y \lrcorner\, d(d\theta) + d(Y \lrcorner\, d\theta) = 0$$
$$\Rightarrow d(Y \lrcorner\, d\theta) = 0 \Rightarrow \exists\, f \in F^0(P) : Y \lrcorner\, d\theta = df \Rightarrow df(X)$$
$$= X(f) = Y \lrcorner\, d\theta(X) = d\theta(Y, X) = -d\theta(X, Y)$$
$$= -X \lrcorner\, d\theta(Y) = 0$$

when restricted to c (formula (9)).

$$\underline{X(f) = 0} \tag{15}$$

Relationship (15) clearly characterises an integral of some system, i.e. a function f defined on phase space P such that f is constant on trajectories.

Remark 5. The conventional approach to Noether's theorem consists in the following statement: If G is any n-parameter symmetry Lie group, i.e. if G leaves the Hamiltonian $H \in F^0(P)$ invariant,

$$\varphi^*(g)H = H \circ \varphi(g) = H \tag{16}$$

there exists n conservation laws. For example the group of rotations $SO(3)$ induces symplectic diffeomorphisms on the phase space $T^*\mathbb{R}^3 \approx \mathbb{R}^6$ which yields the angular momentum $\vec{L}$ to be conserved. The relationship (16) is true iff

$$L_Y H = Y(H) = 0 \qquad \forall Y \in d\varphi_g(\mathfrak{g}) \subset \mathfrak{X}(P) \tag{17}$$

It is known (Hermann, 1968) that the Hamilton-Jacobi theory can be regarded as the study of the characteristic curves and the maximal integral manifolds of the Hamilton form

$$\omega = dp_i\, dq^i - dH.\, dt = d\theta(L) \tag{18}$$

This amounts to saying that a geometric interpretation of the Hamilton-Jacobi partial differential equation can be obtained in terms of the Cartan field $\theta = p_i\, dq^i - H\, .\, dt$ and the following

Proposition 3. If $S(q, t)$ is a solution of the Hamilton-Jacobi equation $(\partial S/\partial t) + H(q_i, \partial S/\partial q_i, t) = 0$ then there is an injection $\psi : N \to T^*M \times \mathbb{R}$ which defines an integral submanifold of the Hamilton form (18). Conversely, if

$$\psi^*\omega = 0 \tag{19}$$

and

$$H^1(M \times \mathbb{R}) = 0 \tag{20}$$

$$\Rightarrow \exists S: \quad \psi^*\theta = dS \tag{21}$$

S is a solution of the Hamilton-Jacobi equation.

Proof. '⇒' By virtue of the adjacent diagram 2, where $(q^1 \dots q^n, \dot{q}^1 \dots \dot{q}^n, t)$ denote coordinates on $TM \times R$, $(q^1 \dots q^n, p^1 \dots p^n, t)$ denote coordinates on $T^*M \times R$, φ is defined on $U \times I$, $U \subset M$ open, by

$$(\pi \times \text{id})\varphi(q,t) = (q,t) \quad \text{and} \quad \psi = \mathscr{L} \circ \varphi$$

$$\begin{matrix} q^i \circ \psi(q,t) = q^i(q,t) \\ p^i \circ \psi(q,t) = p^i(q,t) \end{matrix} \Leftrightarrow \begin{matrix} \psi^*(q_i) = q_i \\ \psi^*(p_i) = p_i \end{matrix} \quad \text{and} \quad t \circ \psi = \psi^* t = t$$

Cartan's form $\theta = p_i dq^i - H.dt$ implies

$$\psi^*\theta = \psi^* p_i \wedge \psi^* dq_i - \psi^* H \wedge \psi^* dt = \frac{\partial S}{\partial q_i} dq_i - \psi^* H \wedge dt$$

$$= dS - \frac{\partial S}{\partial t} dt - H \circ \psi dt = dS - \left(\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t \right) \right) dt$$

$$\Rightarrow \psi^*\theta = dS \Rightarrow \psi^* d\theta = \psi^*\omega = 0$$

i.e. (ψ, N) is an integral manifold of ω.

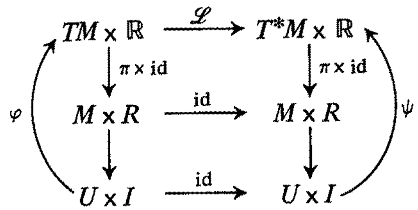

Figure 2.

The converse problem amounts to showing the relationships

$$p_i(q^i, t) = \frac{\partial S}{\partial q_i} \quad \text{and} \quad \frac{\partial S}{\partial t} + H \circ \psi = 0 \tag{22}$$

to hold. In fact

$$\psi^* d\theta = d\psi^*\theta = d\alpha = 0, \qquad H^1(U \times I) = 0 \Rightarrow \exists S : \alpha = dS \qquad \alpha \in \mathring{F}^1$$

$$\Rightarrow \psi^*\theta = dS = \frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial t} dt = \sum p_i dq_i - H \circ \psi \, dt$$

Discussion. In our quest for an integral submanifold map $N \subset T^*M \times \mathbb{R}$, i.e. $\psi : N \to T^*M \times \mathbb{R}$ of the form $\omega = dp_i \wedge dq^i - dH \wedge dt$ we find that

(a) N is the submanifold defined by S = constant and that
(b) the physical component of the Cartan field (θ, c_1) is determined by the cohomology group H^1. According to our introductory remark 2 this is just another way of relating the Cartan–Euler field to the differential geometric structure of P, i.e.

$$\left. \begin{matrix} H^1(U \times I) = 0 \\ \psi : N \xrightarrow{\text{injection}} T^*M \times \mathbb{R} \end{matrix} \right\} \Rightarrow \psi^*\theta = dS \tag{23}$$

3. *Aharanow-Bohm Fields*

Aharanow-Bohm fields (hereafter referred to as AB fields) which are related to the AB effect (v. Westenholz, 1973; Aharanow & Bohm, 1959), display significant topological features and are therefore well-suited to the aim of treating interacting fields of the type (1) in a unified way (cf. Section 6). The topological character underlying the AB effect is due to the following facts.

(1) In the idealised AB effect (cf. adjacent Fig. 3), the path dependence of the wave functions, which are solutions to Schrödinger's equation in the presence of a magnetic field, is given by

$$\psi_i(\vec{x}) = \psi_i^0(\vec{x}) \exp\left(ie \int_{\text{path}\, i} \vec{A} . d\vec{x} \right) := \psi_i^0(\vec{x}) \exp\left(ie \int_{\sigma_1} \omega^1 \right) \qquad i = 1, 2 \tag{24}$$

($\psi_{1,2}^0$ denote the wave functions for the upper and lower beams, respectively, in the absence of a magnetic field in the solenoid). Interference patterns are created which depend on the integral $\oint \vec{A} . d\vec{x} = \int_{c_1} \omega^1$ around the closed circuit $c_1 \in \mathring{C}_1(\mathbb{R}^2 - D_r)$ (space of 1-cycles over $\mathbb{R}^2 - D_r$, the plane minus a disk D_r; the z-direction is dropped along the axis of the solenoid), when the beams are recombined at P. c_1 may be written formally as a linear combination $c_1 = \lambda_1 \sigma_1 + \lambda_2 \sigma_2, \lambda_i \in \mathbb{R}$.

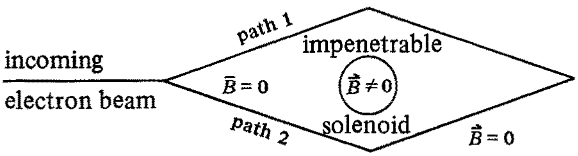

Figure 3.

(2) On the multiply connected physical space $\mathbb{R}^2 - D_r$ of the AB effect the Hamiltonian is not essentially self-adjoint and therefore does not serve to define the dynamics. Therefore, the mathematical model of the corresponding configuration space is necessarily topologically different from $\mathbb{R}^2 - D_r$.

These aspects regarding the AB effect are well-suited to dealing with path-dependent field quantities, also in quantised theories. Mandelstam (1962) has shown that, within such a scheme, Quantum Electrodynamics can be formulated without unphysical states and indefinite metric.

From physical arguments, AB fields are introduced as fields of the type (1) within a topological model which consists of (v. Westenholz, 1973):

(II)

(1) a configuration space M which corresponds to a dynamical AB system;

(2) an AB field and its dual field, given by

$$\omega^2 = E_i \, dx^i \wedge dx^0 + {}^*H_{ij} \, dx^i \wedge dx^j \tag{25}$$

and

$$^*\omega^2 = H_i \, dx^i \wedge dx^0 + {}^*E_{ij} \, dx^i \wedge dx^j$$

and

(3) the simply connected cotangent bundle T^*M over M (which accounts for the quantum mechanical properties of AB fields) and the canonical structure on T^*M.

The AB field (25) is characterised as follows:

Definition 4. An AB field is a pairing

$$(\omega^1, c_1) \tag{27}$$

where $\omega^1 = \Sigma_\mu A_\mu \, dx^\mu$, $x^0 = \text{const.}$ $\mu \in \{0, 1, 2, 3\}$, $c_1 \in \mathring{C}_1(\mathbb{R}^2 - D_r)$, such that

$$\omega^2 = \sum F_{\mu\nu} \, dx^\mu dx^\nu \begin{cases} = 0 & \text{if } {}^*H_{ij} \, dx^i \wedge dx^j \in F^2(\mathbb{R}^2 - D_r) \\ \neq 0 & \text{if supp } ({}^*H_{ij} \, dx^i dx^j) \subset D_r \end{cases} \tag{28}$$

$x^0 = \text{const.}$

$$\omega^2 = d\omega^1 \tag{29}$$

Remark 6. The dual AB field will be specified in Section 6 only.

As regards the canonical structure on the cotangent bundle, it is given in terms of the diffeomorphism

$$\varphi : T^*M \to T^*M : p \to p + \frac{e}{c} A \tag{30}$$

The corresponding canonical system accounts for a canonical formulation of the dynamics of charged particles in terms of the symplectic structure

$$(T^*M, d\bar{\theta}), \qquad \text{where } \bar{\theta} = \bar{p} \, . \, dq, \qquad \bar{p} = p + \frac{e}{c} A \tag{31}$$

AB fields are of particular interest, since, within framework I of Section 1, they can be introduced to explain the virtual quanta of electromagnetic interactions (cf. Section 6). Moreover, AB fields provide, within framework II, a consistent model for the AB effect. This issue will be summarised only (cf. v. Westenholz, 1973). AB fields enjoy the following properties:

Property 1. Let $H_1(\mathbb{R}^2 - D_r)$ and $H^1(\mathbb{R}^2 - D_r)$ denote the first homology and cohomology groups of $\mathbb{R}^2 - D_r$ respectively. Then, by virtue of de Rham's first theorem, there exists a non-degenerate bilinear mapping

$$\beta : H^1(\mathbb{R}^2 - D_r) \times H_1(\mathbb{R}^2 - D_r) \to \mathbb{R}_1$$

$$(\omega^1, c_1) \to \int_{c_1} \omega^1 = \frac{\Delta S}{h} = \Delta\vartheta \tag{32}$$

which assigns the phase shift $\Delta\vartheta$ to the AB field (ω^1, c_1).

Property 2. The gauge transformation of AB fields, $A'_\mu = A_\mu + \partial_\mu \vartheta(x)$ is given in terms of

$$\omega^1, \omega^{1'} \in \{\omega\} \in H^1(\mathbb{R}^2 \setminus D_r) \Leftrightarrow \omega' = \omega + d\vartheta \qquad \text{with } d\omega = d\omega' = 0 \tag{33}$$

such that

$$\int_c \omega' = \int_c \omega + \int d\vartheta \Rightarrow \int_c \omega' = \int_c \omega \tag{34}$$

holds. The gauge transformation property leads automatically to cohomology, which is clearly not the case for any Maxwell field. Therefore, formula (33) is just as good a definition of an AB field as Definition 4.

Property 3. AB fields are 'quantised' fields by virtue of

$$\tilde{\omega}^1 = \frac{\hbar}{2i}\left(\frac{d\Psi}{\Psi} - \frac{d\psi}{\psi}\right) = d\tilde{S} \in F^1(T^*M) \qquad \text{where } \psi = \psi_0\, e^{i(\tilde{S}/\hbar)}, \quad \Psi = \psi_0\, e^{-i(\tilde{S}/\hbar)} \tag{35}$$

This property characterises the AB effect as a quantum effect on a simply connected cotangent bundle T^*M.

Property 4. On T^*M the following holds (units $\hbar = c = 1$):

$$\tilde{\omega}^1 = \tilde{A}_k d\tilde{x}^k = d\tilde{S} \tag{36}$$

We have thus proven the following

Proposition 4 (*v. Westenholz*, 1973). The AB effect may be described in terms of the following topological conditions: There exist fields of the type (1) which satisfy the properties:

(1) $H^1(N') \times H_1(N') \to \mathbb{R} : (\omega^1, c_1) \to \int_{c_1} \omega^1 = \frac{\Delta S}{\hbar} \quad N' = \mathbb{R}^2 - D_r$

(2) $\omega^1, \omega^{1'} \in \{\omega\} \in H^1(N') \Leftrightarrow \omega^{1'} = \omega^1 + d\vartheta \qquad d\omega = d\omega' = 0$

(3) $\tilde{\omega}^1 = \frac{e}{c}\tilde{A}_k d\tilde{q}_k = d\tilde{S}$

(4) $d\tilde{S} = \frac{\hbar}{2i}(d\Psi/\Psi - d\psi/\psi)$

These fields describe the AB effect as quantum effect by virtue of the following statements:

(5) The functions which belong to the quantisation condition (4) are of the form

$$\psi = \psi_0 \exp\left(i\frac{\tilde{S}}{\hbar}\right) \in F^0(\Pi(T_E^*M))$$

and the Bohr-Sommerfeld rules are associated with the energy level constraint surfaces

$$d\bar{S}(\Pi(T_E^*M)) \subset T_E^*M \tag{37}$$

such that

(6) the condition

$$\int_{\varphi_*\tilde{c}} d\bar{S} = \int_{\tilde{c}} dS = n\hbar$$

holds.

Discussion. On T^*M the Hamiltonian will be essentially self-adjoint. The foregoing study exhibits the existence of physical principles underlying the necessity for making the field variables of a gauge-independent theory path-dependent (Mardelstam, 1962):

$$\phi(x, \gamma) = \varphi(x) \exp\left(-ie \int_{\gamma}^{x} dx^{\mu} A_{\mu}\right) \qquad \gamma : [0, 1] \to M^4 \text{ is differentiable}$$

$$= \varphi(x) \exp\left(-ie \int_{c_1} \omega^1\right) \tag{38}$$

These quantities do not depend on the gauge selected for A_μ, i.e. the transformations

$$\varphi \to \varphi\, e^{ie\vartheta(x)}, \qquad \bar{\varphi} \to \bar{\varphi}\,.\, e^{-ie\vartheta(x)}$$

$$A_\mu \to A_\mu + \frac{\partial \vartheta}{\partial x^\mu} \tag{39}$$

leave the matter field variables (38) unaltered, since

$$\phi'(x, \gamma) = \varphi'(x) \exp\left(-ie \int_{c_1} \omega^{1'}\right) = \varphi'(x) \exp\left(-ie \int_{c_1} (\omega^1 + d\theta)\right)$$

$$= \varphi'(x) \exp\left(-ie \int_{c_1} \omega^1 - ie\vartheta(x)\right)$$

Since

$$\int_{\gamma}^{x} d\vartheta = \vartheta(x) \qquad \therefore \quad \phi'(x, \gamma) = \varphi'(x) \exp\left(-ie \int_{c_1} \omega^1\right) \exp\left(-ie\vartheta(x)\right)$$

$$= \varphi(x) \exp\left(-ie \int_{c_1} \omega^1\right) = \phi(x, \gamma)$$

As far as the electromagnetic field variables are concerned, the appropriate gauge-invariant quantities are the electromagnetic field strengths $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Derivatives of $\phi(x, \gamma)$ correspond to the 'gauge invariant derivatives' of $\varphi(x)$

$$\partial_\mu \phi(x, \gamma) = \left[\left(\frac{\partial}{\partial x^\mu} - ieA_\mu\right)\varphi(x)\right] \exp\left(-ie \int_{c_1} \omega^1\right) \tag{40}$$

where

$$[\partial_\mu, \partial_\nu]\phi(x, \gamma) = -ieF_{\mu\nu}(x)\phi(x, \gamma) \qquad c_1 \in C_1(M) \tag{41}$$

To summarise: A description of charged particles interacting with the electromagnetic field within the AB-Mandelstam framework enjoys the property that

> path dependence of the field variables amounts, by virtue of (41), to saying that in the presence of an electromagnetic field the space appears to charged particles as curved.

4. *Yang–Mills Fields*

A rigorous description of interacting fields can be obtained by means of a formal correspondence principle between the Yang–Mills theory (hereafter referred to as YM) and some principal fibre bundle $P(M, G)$ over space-time M with structure group G. A motivation for such an approach where interactions manifest themselves through curvature properties of a bundle is given mainly by Einstein's geometrical description, which uses the curvature of the 'external space' M^4 to describe relativity theory and partly by Mandelstam's approach to field theory as described in our Section 3.

The conventional phenomenological approach to YM fields is the following. The elementary particle fields, which occur in multiplets, are subject to a transformation law

$$\psi'_\alpha(x) = U_\alpha{}^\beta \psi_\beta(x) \tag{42}$$

in the internal space. Yang and Mills considered multiplet fields $\psi(x^\mu)$, $x^\mu \in M$, subject to gauge transformations $U(x^\mu)$ belonging to some gauge group G. Consequently, the extended gauge transformation law is given by

$$\psi'_\alpha(x) = U_\alpha{}^\beta(x)\psi_\beta(x) \tag{42'}$$

If some Lagrangian is invariant under equation (42), the requirement of invariance under the wider transformations (42′) necessitates the introduction of a new field B_μ, called YM-potential, subject to the gauge transformations of the second kind. This field must be coupled to the matter field ψ_α only through the replacement $\partial_\mu \psi_\alpha \to (\partial_\mu - ieB_\mu)\psi_\alpha(x)$.

This issue has been formulated in a more general fashion by Utiyama (1956), whose approach is the following: Consider a system of fields $\psi^{(i)}(x)$ which is invariant under some n-dimensional transformation group G. Suppose G to be replaced by a wider group G', derived by replacing the n parameters by a set of arbitrary functions of $x \in M$. Then the following problem arises:

(1) What kind of field, $A(x)$, is introduced on account of the invariance?
(2) How is the new field transformed under G'?
(3) What form does the interaction between the field A and the original field ψ take, i.e. how can one determine the new Lagrangian $L_1(\psi, A)$ from the original one, i.e. from $L(\psi)$?
(4) What kind of field equation for $A(x)$ are allowable?

More specifically: Consider a Dirac spinor field which interacts with an external electromagnetic field. The total Lagrangian takes the form

$$L = L_0(\psi) + L_0(A_\mu) + L_1(\psi, A_\mu) \tag{43}$$

where $L_0(A_\mu)$ is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \vartheta \tag{44}$$

and $L_0(\psi)$ is invariant under the transformations

$$\psi \to e^{ie\vartheta}\psi \tag{45}$$

If we now require that L be invariant under equation (44) and the gauge transformations $\psi \to e^{ie\vartheta(x)}\psi$ simultaneously, $L_1(\psi, A_\mu)$ is determined uniquely provided we replace in $L_0(\psi)$ the differential operator ∂_μ by the gauge invariant derivative

$$\partial_\mu \to \partial_\mu - ieA_\mu \qquad \text{(cf. equation (40), Section 3)} \tag{46}$$

The interaction term is found to be

$$L_1 = e\,.\,\bar{\psi}\gamma^\mu\psi A_\mu = j^\mu . A_\mu \tag{47}$$

describing the influence on the Dirac field $\psi(x)$ exerted by the electromagnetic field $A_\mu(x)$. The coupled field equations are:

$$[\gamma^\mu(\partial_\mu - ieA_\mu)]\,\psi = m\psi \tag{48}$$

and

$$\Box A^\mu = F^{\mu\nu}_{,\nu} = 4\pi j^\mu \tag{49}$$

where $\partial_\mu j^\mu = 0$.

Summary. The form of the interaction has been determined by the requirement of gauge invariance of the second kind, or, more generally, the form of the interaction between some fields can be determined by postulating invariance under a certain group.

Remark 7. The Lagrangian (47) expresses the principle of minimal electromagnetic interaction, which states that all charged particles have only current type interactions with the electromagnetic field A_μ.

The aim of our Section 4 is to provide a rigorous description of Utiyama's approach.

Let $P(M, G)$ be a principal bundle over space-time $M = M^4$ with structure group G and connection form $\tilde{\omega}^1$. The group G is regarded as the gauge group (YM group). For any non-vanishing cross-section $s: M \to P$ we introduce the quantities

$$\omega^1 = A = s^*\tilde{\omega}^1 \in F^1(M) \tag{50}$$

(s^* denotes the 'pull-back' mapping to s)

$$\omega^2 = B = s^*\Omega^2 \in F^2(M) \tag{51}$$

where

$$\Omega^2 = d\omega + \tfrac{1}{2}[\omega, \omega] \tag{52}$$

denotes the curvature form of ω.

Formulae (50) and (51) represent the YM-potential and YM field respectively. In local coordinates (x^μ) we have

$$\omega^1 = \sum A_\mu dx^\mu \tag{50'}$$

and

$$\omega^2 = \sum B_{\mu\nu} dx^\mu dx^\nu \tag{51'}$$

where

$$B^\alpha_{\mu\nu} = \frac{\partial A_\mu{}^\alpha}{\partial x^\nu} - \frac{\partial A_\nu{}^\alpha}{\partial x^\mu} - c^\alpha_{\rho\sigma}(A_\mu{}^\rho A_\nu{}^\sigma - A_\nu{}^\rho A_\mu{}^\sigma) \tag{53}$$

denote the components of the YM field (51$'$) ($c^\alpha_{\rho\sigma}$ are the structure constants of the holonomy algebra (cf. Proposition 5 below).

To establish a formal connection between the YM description (in the sense of Utiyama's programme) and the fibre bundle approach, we introduce a topological YM field, which is of the type (1), as follows:

$$(\omega^1, c_1)\omega^1 = \sum A_\mu dx^\mu, \qquad c_1 \in C_{x_0} = \{\gamma | \gamma(0) = \gamma(1) = x_0\}$$

$$\text{(set of loops with base point } x_0 \in M) \tag{54}$$

The principle of minimal interaction can now be expressed as follows: Consider the fibre bundle $E(M, F, G, P) = E$ associated with P with standard fibre F. On the product manifold $P \times F$, we let G act differentiably on the right by $(p, f)g = (pg, g^{-1}f)$ and let the quotient space of $P \times F$ by this group action be

$$E = (P \times F)/G \tag{55}$$

Furthermore, let $X = (\partial/\partial x^\mu) \in \mathfrak{X}(M)$ be a C^∞-vector field on M and $\psi \in \Gamma(E)$ a cross-section of E. Then the covariant derivative $\nabla_X\psi$ of ψ in the direction of X can be written as (v. Westenholz, 1972; Trautman, 1968)

$$\nabla_X \psi = \nabla_{\partial/\partial x^\mu} . \psi = \left(\frac{\partial}{\partial x^\mu} - ieA_\mu \right) \psi \equiv \nabla_\mu \psi \tag{56}$$

$$\nabla_\mu \psi_\alpha = \partial_\mu \psi_\alpha - \omega^\beta_{\mu\alpha} \psi_\beta \tag{56'}$$

(cf. v. Westenholz, 1972; Trautman, 1968; Hiley & Stuart, 1971)

This is the familiar rule of 'minimal interaction', i.e. relationship (56) may be used as a basis for introducing interactions between charged particles and fields of the electromagnetic type (i.e. also YM-potentials).

Remark 8. The gauge covariant derivative of a matter field (56′) may be used to define the quantities

$$d\psi_\alpha = dx^\mu . \nabla_\mu \psi_\alpha = \omega^\beta_{\alpha\mu} \psi_\beta dx^\mu \tag{57}$$

which constitute the difference between the field values of a multiplet that has been displaced in a *parallel* fashion from x^μ to $x^\mu + dx^\mu$. That is, the gauge potential ω^1 provides a gauge invariant definition of equivalent, i.e. parallel, multiplets at neighbouring events in terms of the relationship

$$\psi_\alpha(x^\mu + dx^\mu) - \psi_\alpha(x^\mu) = \omega^\beta_{\alpha\mu} \psi_\beta dx^\mu \tag{58}$$

Remark 9. Formula (56) is related to the horizontal lift of the vector field $X = (\partial/\partial x^\mu) \in \mathfrak{X}(M^4)$ to $P(M^4, G)$ in the following way: Consider the canonical basis $(\epsilon_\mu, \epsilon_\nu{}^x) = (\epsilon_\alpha)$ of the tangent space $T_p(P(M^4))$ to $p \in P$. Clearly $d\pi\epsilon_\mu = e_\mu \equiv (\partial/\partial x^\mu)$, $\{e_i\}$ is the canonical basis of $T_x(M)$ and $d\pi\epsilon_\nu{}^x = 0$, where the projection π is given in local coordinates by $(x_\mu, a_\mu{}^x) \to x^\mu$. A basis of the horizontal space H_p is then given in terms of

$$\tilde{X}_\mu = \sigma_\mu{}^\alpha \epsilon_\alpha = \epsilon_\mu - \Gamma^\nu_{\mu x} \epsilon_\nu{}^x = \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu x} a_\lambda{}^x \frac{\partial}{\partial q_{\lambda\mu}} \tag{59}$$

and this is the horizontal lift of $\partial/\partial x^\mu$, since

$$d\pi\tilde{X}_\mu = d\pi\epsilon_\mu - d\pi(\Gamma^\nu_{\mu x} \epsilon_\nu{}^x) = e_\mu - \Gamma^x_{\mu\nu} d\pi(\epsilon_\nu{}^x) = e_\mu = \frac{\partial}{\partial x^\mu}$$

It can be shown (Hiley & Stuart, 1971) that (59) takes the form

$$\tilde{X}_\mu = \frac{\partial}{\partial x^\mu} - ieA_\mu{}^\rho E_\rho \tag{60}$$

where $\{E_\rho\}$ is a basis of the Lie algebra $\mathfrak{g}$ of G. Therefore, roughly speaking, it turns out that covariant differentiation corresponds to Lie differentiation as displayed by Proposition 1.3, p. 116, of Kobayashi & Nomizu (1963).

Remark 10. Within the correspondence principle between the YM theory and the fibre bundle approach, it turns out that the gauge invariant derivative (formulae (40) and (46)) is nothing more than the generalised covariant derivative in the bundle E.

As regards the matter field which interacts by means of (56) with the electromagnetic field, it is defined by the transformation law

$$\psi'(x) = U(x)\psi(x) \tag{61}$$

where $U(x) = \rho(g^{-1})$ for some $g \in G$, that is, $U(x)$ is an element of $\rho(G)$, the representation of G, on the standard fibre F; $U: M \to \rho(G)$.

Example 1. Let M be space-time, $G = SO(2)$, i.e. $P = P(M, SO(2))$ and let $\rho : SO(2) \to U(1) = \{e^{i\alpha}, \alpha \bmod 2\pi\}$, i.e. $SO(2)$ acts on the standard fibre $\mathbb{C}$ of the bundle $E = P \times \mathbb{C}/SO(2)$ associated with P. The gauge transformations of the first kind are given by $U: M \to U(1)$, $U(x) = e^{ie\vartheta(x)}$. They act in the set of cross-sections of E by

$$\psi'(x) = e^{-ie\vartheta(x)}\psi(x) \tag{61'}$$

The field ψ can therefore be interpreted as a matter field which describes charged particles. Since the gauge group is a one-parameter Abelian group, $c^{\alpha}_{\rho\sigma} \equiv 0$ and relationship (53) becomes (53′), $B_{\mu\nu} = F_{\mu\nu} = (\partial A_\mu/\partial x^\nu) - (\partial A_\nu/\partial x^\mu)$, which is just the electromagnetic field tensor.

Remark 11. By virtue of this example, electrodynamics can be regarded as the theory of an infinitesimal connection in a principal fibre bundle with structure group $SO(2)$ as pointed out by Trautman (1968).

Example 2. Let M be space-time and $G = SU(3)$. The fundamental representations of $SU(3)$ are $D^3(1, 0)$ and $D^3(0, 1)$ and

$$D^3(1, 0) \otimes D^3(0, 1) = D^8(1, 1) \oplus D^1(0, 0),$$

where $D^8(1, 1)$ is the eight-dimensional adjoint representation of $SU(3)$. For the corresponding spinor field $\psi : M \to \mathbb{C}^4$, which classifies, say baryons, we have:

$$\psi'_\alpha(x) = \sum_{\beta=1}^{\infty} U_\alpha{}^\beta(x)\psi_\beta(x), \qquad U_\alpha{}^\beta \in Ad(SU(3)) \cong D^8(1, 1) \tag{61''}$$

Remark 12. The Ambrose–Singer Theorem (Ambrose & Singer, 1953) proves that the curvature form of the bundle connection spans the Lie algebra of the restricted holonomy group ϕ_{x_0}, i.e.

$$U_\alpha{}^\beta = \delta_\alpha{}^\beta + \tilde{\Omega}^\beta_{\alpha\mu\nu}\, dx^\mu\, dx^\nu \tag{62}$$

where $(\tilde{\Omega}_\alpha{}^\beta)$ denotes the matrix of the curvature form Ω^2 and

$$\Omega = \sum_{\alpha,\beta} \Omega_\alpha{}^\beta E_\beta{}^\alpha; \qquad \{E_\alpha{}^\beta : \alpha, \beta = 1 \ldots m\} \tag{63}$$

constitutes a basis of $\mathfrak{g}$.

In the case of Example 2, where

$$G = SU(3) \tag{64}$$

such a basis is given by $\lambda_\alpha{}^\beta$, which are the eight traceless Hermitean Gell-Mann matrices.

Proposition 5 (*v. Westenholz*, 1972). Let $P(M^4, G)$ be the principal bundle over space-time with structure group G, $\tilde{\omega}^1$ its connection 1-form and Ω^2 the curvature form of $\tilde{\omega}^1$.

Then the following holds: (1) The fields

$$(\omega^1, c_1), \qquad \omega^1 = A = \sum A_\mu \, dx^\mu \in F^1(M^4),$$

$$c_1 \in C_{x_0} = \{\gamma | \gamma(1) = \gamma(0) = x_0\} \tag{66}$$

and

$$(\omega^2, c_2), \qquad \omega^2 = B = \sum B_{\mu\nu} \, dx^\mu \, dx^\nu \tag{67}$$

can be regarded as being derived from the structure P by virtue of $s^*\tilde{\omega}^1 = \omega^1$ and $s^*\tilde{\omega}^2 = s^*\Omega^2 = \omega^2$, $s: M \to P(M^4)$ is a cross-section. (2) The YM field (66) interacts with a matter field of charged particles,

$$\psi'_\alpha(x^\mu) = U_\alpha{}^\beta(x^\mu)\psi_\beta(x^\mu) \tag{42}$$

by virtue of

$$\nabla_\mu \psi = \left(\frac{\partial}{\partial x^\mu} - ieA_\mu \right) \psi \tag{56}$$

which is the principle of minimal coupling. (3) The topological field components $c_1{}^i \in \overset{\circ}{C}_1(M)$ of the YM field (66) determine completely the interaction symmetry.

Proof. With each loop $\gamma_i = c_i{}^1 \in C_{x_0}$ is associated the parallel displacement

$$\gamma_i \to \tau_{\gamma_i}: \pi^{-1}(x_0) \to \pi^{-1}(x_0), \qquad \tau_{\gamma_i}(pg) = \tau_{\gamma_i}(p) g \; \forall g \in G, \qquad p \in P \tag{68}$$

These automorphisms $\{\tau_{\gamma_i}\}$ of the fibre $F_{x_0} = \pi^{-1}(x_0)$ can be shown to be a group, the holonomy group of the connection (v. Westenholz, 1972). The unique horizontal lift of the differentiable path $\gamma: [0, 1] \to M^4$ beginning at $p \in \pi^{-1}(x_0)$ is the integral curve of $\tilde{X}_\mu$, i.e. we have

$$\tilde{\gamma} \to \tilde{X}_\mu(p) = \frac{\partial}{\partial x^\mu} - ieA_\mu{}^\rho E_\rho \qquad \text{(cf. (67))} \tag{69}$$

Therefore, the gauge-invariant derivative (56) is determined by the field components $c_1{}^i \in C_1{}^0(M^4)$ of (66). These field components obviously also determine the transformations of ϕ_{x_0}. ϕ_{x_0} in turn characterises the minimal interaction through (60) and (62).

Remark 13. Formulae (68) and (69) display how the YM fields, (66) and (67), are associated with the geometrical structure of $P(M, G)$, i.e. the interaction with the matter field (42) is mediated through the topological components $c_1{}^i$. On the other hand, if the matter field is parallely transferred around some loop $\gamma \in C_{x_0}$ (cf. Remark 8) then upon return to x_0 it will, in general, differ from its original value due to the non-integrability of the bundle connection, i.e. by virtue of

$$\nabla_\mu \nabla_\nu \psi_\alpha - \nabla_\nu \nabla_\mu \psi_\alpha = \tilde{\Omega}^\beta_{\alpha\mu\nu} \psi_\beta \tag{70}$$

that is

$$[\nabla_\nu, \nabla_\mu] = \left[\frac{\partial A_\mu{}^\alpha}{\partial x^\nu} - \frac{\partial A_\nu{}^\alpha}{\partial x^\mu} - C^\alpha_{\rho\sigma}(A_\mu{}^\sigma A_\nu{}^\rho - A_\nu{}^\rho A_\mu{}^\sigma) \right] E_\alpha = B^\alpha_{\mu\nu} E_\alpha \tag{71}$$

This operation of parallel displacement around γ induces the linear transformation

$$\psi'_\alpha(x^\mu) = U_\alpha{}^\beta(x^\mu)\psi_\beta(x^\mu) \tag{42}$$

This transformation law defines a quark. Since (42) is induced by the 'homologous' YM field component (66) it is quite natural to interpret a quark as an elementary loop $c_1{}^i \in C_1{}^0$ of a YM field. In particular, quarks may be viewed as quantised elementary loops (Jehle, 1971), but only when interlinked with other loops (cf. Section 6).

Remark 14. If the restricted holonomy group ϕ_{x_0} is given, one can always determine the topological field component $c_i{}^1 = \gamma_i$ of a YM field $(\omega^1, c_i{}^1)$ such that $\gamma_i \to \tau_{\gamma i} \in \phi_{x_0}$.

Remark 15. The local symmetry group ϕ_{x_0} is an internal symmetry. A necessary condition for the minimal coupling to be 'switched off' is

$$\phi_{x_0} = I \tag{72}$$

i.e. the local symmetry is trivial (refer to Section 6).

In order to achieve this rigorous approach to Utiyama's programme it remains to find the field equations. These are

$$d^*B = 4\pi\omega^3 \tag{73}$$

the YM equations, where $^*B$ is the field which derives from the dual curvature form $^*\Omega$ and

$$\omega^3 = j_0 dx^1 dx^2 dx^3 + \ldots + j_3 dx^0 dx^1 dx^2 \tag{74}$$

denotes the 3-form which stands for the conserved current

$$j_\mu = \partial_\mu \left(\frac{\partial L}{\partial \psi_\alpha} \right) \gamma_\alpha \tag{75}$$

In particular, if $G = SO(2)$, equation (73) reduces to Maxwell's equations

$$d^*\omega^2 = 4\pi j, \qquad j^\mu = \left(\rho, \frac{1}{c}\vec{\imath}\right), \qquad {}^*F_{\mu\nu} = \frac{1}{2!}\,\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{76}$$

In order to reformulate, within this framework, both coupled field equations (48)-(49) one must have recourse to an appropriate variational principle. This will be published elsewhere.

5. *A Differential Topological Approach to Noether's Theorem: The Topological Gell-Mann Current Field*

Noether's theorem, connecting one-parameter groups of symmetries of a variational problem and conserved currents, is fundamental in field theory. The adequate mathematical approach to the study of symmetry properties of a system is given in terms of groups of automorphisms, i.e. of transformations which carry over the system into itself. If a system is invariant under a certain group of transformations then from this symmetry property there follows the conservation of certain dynamical observables of this system.

In the phenomenological Lagrangian formulation such conservation laws are obtained as a consequence of the transformation properties which some Lagrangian must undergo, namely that it transforms like a scalar under some transformation group:

$$L(\tilde{\psi}(\tilde{x}),\ \tilde{\psi}_{,\mu}(\tilde{x})) = L(\psi(x),\ \psi_{,\mu}(x)) \tag{77}$$

As a simple example suppose the transformations

$$\psi_\alpha \to e^{i\vartheta}\psi_\alpha \tag{78}$$

and

$$\Psi_\alpha \to e^{-i\vartheta}\Psi_\alpha \tag{79}$$

to leave unchanged the Lagrangian $L = L(\psi, \partial_\mu\psi)$. It follows that

$$j^\mu(x) = i\vartheta\left(\frac{\partial L}{\partial\overline{\psi}_{\alpha,\mu}}\,\Psi_\alpha - \frac{\partial L}{\partial\psi_{\alpha,\mu}}\,\psi_\alpha\right) \tag{80}$$

is the conserved electric charge current density, i.e.

$$\partial_\mu j^\mu = 0 \tag{81}$$

Equation (80) then defines a constant of motion, which is given to be the total charge

$$Q = \int_{x^0=\mathrm{const}} j^0(x)\,d^3x \tag{82}$$

in agreement with the one-dimensional Abelian Lie group (78), i.e. (79), and thus satisfying Noether's theorem.

Our subsequent approach to Noether's theorem is aimed at a differential topological version of the basic statements relating properties of invariance to conservation laws.

Remark 16. Our differential topological approach to Noether's theorem for systems of infinite degrees of freedom is modelled after a modified version of Proposition 2, i.e. Noether's theorem for finite degree systems:

Proposition 6 (*v. Westenholz*, 1973). Every infinitesimal symmetry of some Lagrange form generates on every state a conserved current.

Proof. Let $X \in \mathfrak{X}(P)$ be an infinitesimal symmetry of a Lagrangian n-form ω, i.e. $X(\omega) = 0$. Set

$$\alpha = X \lrcorner\, \omega \tag{83}$$

for any state c (Definition 1)

$$\Rightarrow d\alpha = d(X \lrcorner\, \omega) = X \lrcorner\, d\omega - X(\omega) = X \lrcorner\, d\omega$$
$$\Rightarrow d\alpha|_c = X \lrcorner\, d\omega|_c = 0 \tag{84}$$

Formula (84) states that (83) may be viewed as a conserved current. Proposition 6 provides an important representation for global observables in terms of the functional

$$\langle c, X \lrcorner\, \omega \rangle = \int_c X \lrcorner\, \omega = f(c, X) \tag{85}$$

Questions related to invariance properties of Lagrangians which characterise systems with infinite degrees of freedom can be associated with jet bundles (Hermann, 1970). We recall the construction of the bundle $J^1(E)$ of 1-jets associated with the fibred manifold $\pi : E \to M$. Consider

$$M \times \Gamma(E) = \{(x, s) : x \in U \subset M,\ s \in \Gamma(E),\ s : U \to E\}$$

and the equivalence relation $\mathcal{R}$, defined by

$$(x, s) = (x', s') \bmod \mathcal{R} \Leftrightarrow D_x s = D_x s' \qquad (D \text{ stands for the derivative})$$

Let $j : M \times \Gamma(E) \to J^1(E) = M \times \Gamma(E)/\mathcal{R}$ be the canonical map of $M \times \Gamma(E)$ on the quotient $M \times \Gamma(E)$ by $\mathcal{R}$. The set $J^1(E)$ admits a natural structure of differentiable manifold, such that the map $\tau : J^1(E) \to E$, $\tau(j(x, s)) = s(x)$ is differentiable and $\pi^1 = \pi \circ \tau : J^1(E) \to M$ is a differentiable bundle. This is summarised in terms of the following diagram:

$$\begin{array}{ccc} J^1(E(M^4)) & \xrightarrow{\ \tau\ } & E(M^4) \\ \Big\downarrow \pi^1 & \swarrow \pi & \\ M^4 & & \end{array}$$

Assume now that M^4 is orientable and let $dp = dx^0 \wedge \ldots \wedge dx^3$ be a nowhere zero 4-form on M^4 and let

$$L : J^1(E(M^4)) \to \mathbb{R}$$
$$(x^\mu, \psi_\alpha, \partial_\mu\psi_\alpha) \to L(x^\mu, \psi_\alpha, \partial_\mu\psi_\alpha) \tag{86}$$

be a real-valued function, the Lagrangian, defined on $J^1(E)$, where a coordinisation is supposed to be given in terms of $(x^\mu, \psi_\alpha, \partial_\mu\psi_\alpha)$. A first approach to Noether's theorem, due to Hermann (1970), can be outlined as follows. Consider the Lagrange 4-form

$$\tilde{\omega}^4 = L\pi^{1*}\, dp \in F^4(J^1(E)) \tag{87}$$

Define an infinitesimal symmetry of L to be a vector field $X \in \mathfrak{X}(E(M^4))$ which has a first-order prolongation $X^1 \in \mathfrak{X}(J^1(E))$ such that

$$X^1(L\pi^{1*}\, dp) = 0 \tag{88}$$

holds. Then Noether's theorem states:

Proposition 7. There exists a vector field $Y \in \mathfrak{X}\,(M^4)$ such that

$$d(Y \lrcorner\, dp) = j^1(s)^* X^1(\tilde{\omega}^4) = 0 \tag{89}$$

provided X^1 constitutes a symmetry field of L. $j^1(s) : M^4 \to J^1(E)$ is a 1-jet of the extremal s.

A proof of this proposition can be found in Hermann (1970).

Remark 17. A vector field on $J^1(E)$ is a first-order prolongation of some $X \in \mathfrak{X}\,(E)$, if $\mathfrak{X}\,(E) \to \mathfrak{X}\,(J^1(E)) : X \to X^1$ is a Lie algebra homomorphism.

By virtue of relationship (89), $Y \lrcorner\, dp$ can be interpreted as a conserved current, i.e.

$$Y \lrcorner\, dp = \omega^3 = j_0\, dx^1 dx^2\, dx^3 + \ldots + j_3\, dx^0\, dx^1\, dx^2 \tag{90}$$

which is just (74). By virtue of (75), $j_\mu = (\partial L/\partial\psi_{\alpha,\mu})\gamma_\alpha(x)$, $\partial^\mu j_\mu = 0$, it turns out that this divergence-free 4-current, which corresponds to (90), entails the condition

$$d\omega^3 = 0 \tag{91}$$

to hold.

An alternate approach to Noether's theorem is given as follows: Let $s : M^4 \to E$ be a section of the bundle $E(M^4)$ and $\bar{\varphi}_t : E \to E$ be a one-parameter family of automorphisms of E such that

$$\pi \circ \bar{\varphi}_t = \varphi_t \circ \pi \tag{92}$$

where $\varphi_t : M^4 \to M^4$ is a family of diffeomorphisms of M^4, whose infinitesimal generator is X_M. Suppose that the families s_t and $\bar{\varphi}_t$ preserve the action

$$I_{\varphi_t(U)}(s_t) = I_U(s) \tag{93}$$

where $I_U(s) = \int_u L(j_1(s))\, dp$. $U \subset M^4$ is a compact set,

$$\varphi_t^*(L(j_1(s_t)\, dp) = L(j_1(s))\, dp \tag{94}$$

where $s_t = \bar{\varphi}_t \circ s \circ \varphi_t^{-1}$.

This leads to the following (cf. v. Westenholz, 1973; Goldschmidt & Sternberg, 1973):

Proposition 8 (*Noether*). Let s be an extremal and suppose that u_t is a one-parameter family of sections of $J^1(E)$ with $u_0 = u = j_1(s)$. $\bar{\varphi}_t$ is a one-parameter family of diffeomorphisms of the fibred manifold $E(M^4)$ which preserves the action (93), i.e. which satisfies (94). Then the 3-form

$$L(j_1(s))X_M \lrcorner\, dp + u^*(X \lrcorner\, \theta) \tag{95}$$

on M^4 is closed, where $X_M = (d\varphi_t/dt)|_{t=0}$ and $X = (du_t/dt)|_{t=0}$.

Proof. Set $u_t = j_1(s_t)$, then by Lemma 1, below, $j_1(s_t)^*\theta = L(j_1(s_t))\, dp = u_t^*\theta$ by (94), $\varphi_t^* u_t^*\theta = u^*\theta$, where $u = j_1(s)$, therefore

$$\begin{aligned}\frac{d}{dt}(\varphi_t^* u_t^*\theta)|_{t=0} &= d\varphi^*(X_M \lrcorner\, u^*\theta) + \varphi^*(X_M \lrcorner\, d(u^*\theta) + \varphi^*[du^*(X \lrcorner\, \theta) \\ &\quad + u^*(X \lrcorner\, d\theta)] \\ &= \varphi^*[d(X_M \lrcorner\, u^*\theta) + (X_M \lrcorner\, u^* d\theta) + du^*(X \lrcorner\, \theta) + u^*(X \lrcorner\, d\theta)] \\ &= 0\end{aligned}$$

$u^*\, d\theta = 0$, since $u^*\, d\theta \in F^5(M^4)$; $u^*(X \lrcorner\, d\theta) = 0$ since $s: M^4 \to E$ is an extremal by assumption.

$$d(X_M \lrcorner\, u^*\theta) + du^*(X \lrcorner\, \theta) = 0 \qquad \text{and} \qquad u^*\theta = L(j_1(s))\, dp \tag{96}$$

Lemma 1. There exists a unique differential form θ of degree 4 on $J^1(E)$ such that

$$j_1(s)^*\theta = L(j_1(s))\, dp \tag{97}$$

for all sections s of $E(M^4)$, where $x \mapsto j_1(s)(x)$ is a differentiable section of $J^1(E)$.

For proof refer to Goldschmidt & Sternberg (1973).

Remark 19. The differential form θ of Lemma 1 generalises, within the framework of jet bundles, the Cartan form (6), $\theta = p_i\, dq^i - H\, .dt$, on the co-tangent bundle (cf. Section 2). Moreover, the relation between this generalised Cartan form and the functional

$$s \to I(s) = \int_u u^*\theta \tag{98}$$

is just a generalisation of the functional (7) of Section 2.

Discussion. The Jet-bundle formalism also fits the principle of minimal interaction (Section 4). Therefore, Noether's theorem can be, in principle, restated for gauge invariant theories, i.e. the theory of YM fields, on account of some modifications. In fact, suppose $E(M^4)$ be the associated vector bundle of some principal bundle $P(M^4, G)$. Let $L : J^1(E) \to \mathbb{R}$ be a Lagrangian on $J^1(E)$. The principle of minimal interaction leads to the following bundle isomorphism

$$\alpha : J^1(E) \to J^1(E) \tag{99}$$

such that

$$\alpha^*(L)(j_1(s)(x)) = L(\alpha(j_1(s)(x)); \quad j_1(s, x) \in J^1(E) \tag{100}$$

where $\alpha^* : F^p(J^1(E)) \to F^p(J^1(E))$ denotes the dual map of the smooth mapping α. In terms of this set-up it can be shown (Hermann, 1970) that the Euler–Lagrange operator associated with the Lagrangian $\alpha^*(L)$ differs from the usual Euler–Lagrange operator by the term $\partial_\mu - ieA_\mu$. That is, if one takes a Lagrangian of the form $L = \beta_{\alpha\mu}(\phi)\phi_{\alpha\mu}$, then

$$\partial_\mu(L_{\alpha\mu}) - L_\alpha = \frac{\partial}{\partial x^\mu}[\beta_{\alpha\mu}(\phi)] - \beta_{\beta\mu,\alpha}(\phi)\partial_\mu\phi_\beta(x) \tag{101}$$

becomes

$$\frac{\partial}{\partial x^\mu}[\beta_{\alpha\mu}(\phi(x))] - \beta_{\beta\mu,\alpha}(\phi(x))](\partial_\mu - ieA_\mu)\phi_\beta(x) \tag{102}$$

We now describe the construction of the conserved observables which are associated with Noether's theorem. Constants of motion, such as the total charge (formula (82)), arise within a differential topological framework in the following way: Let $c_3 \subset M^4$, $c_3 \in \mathring{C}_3(M^4)$ be a three-dimensional submanifold of M^4. Then, according to de Rham's theorem, there is a non-degenerate bilinear mapping

$$\beta : H^3(M) \times H_3(M) \to \mathbb{R} \tag{103}$$

$$(\omega^3, c_3) \to f(s, c_3, Y) = \int_{c_3} \omega^3 = \int_{c_3} Y \lrcorner\, dp \tag{104}$$

Definition 5. The pairing (103), (ω^3, c_3), $\omega^3 \in \mathring{F}^3(M^4)$, $c_3 \in \mathring{C}_3(M^4)$, is called a topological Gell-Mann current field. The quantity (104), $f = f(s, c_3, Y)$, denotes an observable which is said to be associated with the Gell-Mann field (ω^3, c_3).

Remark 20. The cohomological component of the Gell-Mann field (103) is the conserved current ω^3 given by formulae (90) and (92).

Remark 21. The Gell-Mann field (103) is a topological field of the type (1).

Proposition 9. Observables which are associated with Gell-Mann current fields are constants of motion.

Proof. Let c_3 and c_3' be submanifolds of M^4 which cobound, i.e., are the boundaries of the four-dimensional region c_4, then

$$\int_{c^3} \omega^3 - \int_{c_3'} \omega^3 = \int_{c_3 - c_3'} \omega^3 = \int_{\partial c_4} \omega^3 = \int_{c_4} d\omega^3 = 0$$
$$\Rightarrow f(s, c_3, Y) = f(s, c_3', Y) \tag{105}$$

Remark 22. The observable (105) is the field theoretic counterpart to the functional (85) of finite degree of freedom systems.

Remark 23. The conservation condition (89) $d(Y \lrcorner dp) = 0$ of Proposition 7 ensures that an observable f of the type (104) does not depend on the choice of the 'Cauchy data' submanifold $c_3 \subset M^4$.

In particular, choose c_3 as the submanifold $x_0 = \text{const.}$, then

$$(\omega, c_3) \to Q = \int_{c_3} \omega_0(t, x)\, dx^1 \wedge dx^2 \wedge dx^3 \tag{106}$$

which is the charge generated by the current ω^3 (cf. (82)). Assignment (106) is to be regarded as a generalised charge. Now, if the Gell-Mann field corresponds to the YM group $SU(3)$ then this generalised charge is given by

$$(\omega^3, c_3) \to Q = e(I_3 + \tfrac{1}{2}Y) \tag{107}$$

(Y denotes the hyper-charge and I_3 the three-component of the isospin).

The Gell-Mann current fields provide a structural interpretation to the currents that arise in the YM theory. Generalised currents that generate the generalised charge (107) are defined by the YM equation (73), $d{*}B = 4\pi\omega^3$, which exhibits how the cohomologous Gell-Mann field component is related to the dual YM field $*B$. Within such a structural analysis of currents it turns out that the currents defined by (73) span the Lie algebra of the holonomy group ϕ_{x_0}, i.e. this Lie algebra consists of all linear combinations of the quantities $\Omega^{\beta}_{\alpha\mu\nu}$, $\nabla_\lambda \Omega^{\beta}_{\alpha\mu\nu}$, $\nabla_\tau \nabla_\lambda \Omega^{\beta}_{\alpha\mu\nu}$. . . . Since forces or interactions are supposed to manifest themselves through the curvature properties of some principal bundle, our approach to (strong) interactions is consistent with the following conventional interpretation, which says:

Fundamental objects for strong interaction physics are not the fields $\psi(x)$ but the currents j^μ which mediate the interactions by virtue of the principle of minimal interaction and

(a) the Gell-Mann current field (ω^3, c_3) in conjunction with
(b) the Yang-Mills equation $d{*}B = 4\pi\omega^3$.

The underlying geometrical structure is a principal bundle of the type $P(M, SU(n))$.

Remark 24. The charge assignment (107) can be understood in terms of the commutators ($\{E_\alpha{}^\beta : \alpha, \beta = 1 \ldots 8\}$ is a basis of $\mathfrak{g}(SU(3))$)

$$[E_\mu{}^\alpha, E_\beta{}^\lambda] = \delta_\beta{}^\alpha E_\mu{}^\lambda - \delta_\mu{}^\lambda E_\beta{}^\alpha \qquad \text{which imply} \qquad \begin{aligned} E_1{}^1 &= I_3 + \tfrac{1}{2}Y \\ E_2{}^2 &= -I_3 + \tfrac{1}{2}Y \\ E_3{}^3 &= -Y \qquad \text{etc.} \end{aligned}$$

6. *Non-Local Field Theoretic Description of Interactions*

The purpose of this section is to study, within the framework of the preceding five sections, in more detail how interactions which correspond to an interaction Lagrangian $L_I(\psi, A_\mu) = ej^\mu A_\mu$ and, more particularly, effects of such interactions, such as 'dressed' and 'bare' charge, virtual quanta (photons) etc., must be described.

What is now the exact meaning of interaction if the fields involved are topological fields of the type (1)? As such fields are related to topology, it seems quite natural to define interaction in terms of the topological linking number of the topological components of the interacting fields. More precisely, let the basic set-up be given in terms of fields $(\omega_1{}^1, c_1{}^1)$, $(\omega_2{}^1, c_1{}^2), \ldots, (\omega_n{}^1, c_1{}^n)$. To start with, we confine ourselves, however, to the case of two fields, say the electromagnetic field $(\omega_1{}^1, c_1{}^1)$, $\omega_1{}^1 = \Sigma_\mu A_\mu . dx^\mu$, which interacts with the matter field $(\omega_2{}^1, c_1{}^2)$ whose topological component $c_1{}^2 \in \mathring{C}_1(M)$ represents a basic unit of particle physics (Remark 13). Then the topological interaction scheme consists of

(Ia) A topological interaction description in terms of the linking number $l(c_1{}^1, c_1{}^2)$ of the quantised loops $c_1{}^1$ and $c_1{}^2$.
(Ib) A 'total' field which accounts for the principle of minimal coupling and the corresponding equations of motion of the fields.
(II) The property of *non-locality*. There is a formal correspondence principle between the non-local AB fields which account for the quantisation of the loops $c_1{}^i$ and the virtual quanta involved.

This topological approach provides an estimate of the electromagnetic interaction constant $\alpha = e^2/\hbar c$.

Remark 25. The principle of minimal interaction introduces a 'total' field in terms of the action integral

$$I = \int j^\mu . A_\mu \, d^4x = \int (eA_\mu + \mu s^\nu F^*_{\nu\mu})\, dx^\mu = \int \omega^1, \ \omega^1 \in F^1(M^4)$$

(Voros, 1972)

Remark 26. Let $(\omega_2{}^1, c_1{}^2)$ represent some matter field. According to Lichnerowicz (1964) one can always construct a spinor field $\psi = S\omega_2{}^1$ which defines the corresponding interaction quark by $\psi'_\alpha = U_\alpha{}^\beta \psi_\beta$ (cf. formula (61)).

Remark 27. All local relativistic field theories with interactions are divergent; 'renormalisability' then expresses the fact that when the observable quantities are re-expressed in terms of the 'renormalised' charge and mass, no divergences appear. In order to circumvent such ill-defined concepts, one might argue that a more correct field theoretic approach to interaction should be non-local in character.

A description of interacting topological fields may now be given as follows. Let $f: S^1 \to \mathbb{R}^3$ be an imbedding such that $f(S^1) = \partial M$ for the compact oriented

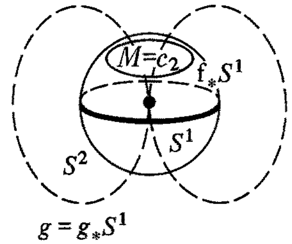

Figure 4

2-manifold with boundary $M = c_2$ (cf. Fig. 4) and $S^1 = \{z : |z| = 1\}$. Let $g : S^1 \to R^3$ be a map such that whenever

$$g(t) = p \in M \tag{108}$$

one has $dg/dt \notin T_p$. f and g are supposed to be C^∞-maps and $g(S^1) \cap f(S^1) = \phi$. Now define the map $\chi_{f,g} \equiv \chi : S^1 \times S^1 \to S^2 \subset R^3 - \{0\}$ by

$$\chi(s, t) = \frac{g(s) - f(t}{\|g(s) - f(t)\|}$$

$= \xi$ and $\deg \chi = l(f, g)$, that is, the degree of the map χ is by definition the linking number of f and g. Then the following holds

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Proposition 10. Suppose $f = f_* S^1$ to be a steady current-carrying loop. Then the circulation around the loop $g = g_* S^1$ is the quantity

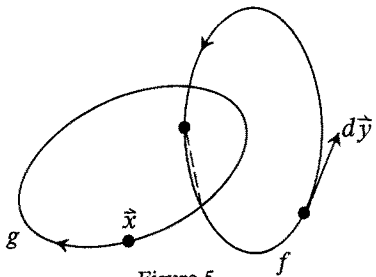

Figure 5

$$l(f, g) = \int_{g_* S^1} \vec{H} . d\vec{x} \tag{109}$$

where

$$\vec{H} = \vec{H}(\vec{x}) = -\frac{I}{4\pi} \int \frac{(\vec{y} - \vec{x}) \times d\vec{y}}{\|\vec{y} - \vec{x}\|^3} \tag{110}$$

denotes the total magnetic field at $\vec{x} \in \mathbb{R}^3$ due to the current $I = -e/\tau$ in f.

Proof. Since $l(f, g) = \deg \chi$ and $\int \chi^*_\omega = \deg \chi \int \omega \int_{S_2} \omega = 4\pi$. We have to show that

$$l(f, g) = \frac{1}{4\pi} \int_{S^1 \times S^1} \chi^* \omega = -\frac{1}{4\pi} \int \frac{A(s, t)}{\|g(s) - f(t)\|^3} ds . dt \tag{111}$$

where

$$A(s, t) = \begin{vmatrix} g_1(s) - f_1(t) & g_2(s) - f_2(t) & g_3(s) - f_3(t) \\ g_1'(s) & g_2'(s) & g_3'(s) \\ f_1'(t) & f_2'(t) & f_3'(t) \end{vmatrix} \tag{112}$$

that is for $i \neq j \neq k$:

$$A(s,t) = -\sum_i (-1)^{i+1}(g_i(s) - f_i(t))[g_j'(s)f_k'(t) - g_k'(s)f_j'(t)] \tag{113}$$

It is readily seen that the right-hand side of (111) equals (109), for

$$y_i = g_i(s); \qquad x_i = f_i(t); \qquad \frac{dy_i}{ds} = g_i'(s); \qquad \frac{dx_i}{dt} = f_i'(t)$$

Now $\omega' = \Sigma_{i=1}^{2}(-1)^{i+1}\xi_i\, d\xi_j\, d\xi_k$ is the element of volume of S^2 and $\omega = \omega'/\|\xi\|^3$ the elements of volume of $R^3 - \{0\}$

$$\Rightarrow \chi^*\omega = \frac{1}{\|\xi(s,t)\|^3}\sum(-1)^{i+1}\xi_i\left(\frac{\partial \xi_j}{\partial s}ds - \frac{\partial \xi_j}{\partial t}dt\right)\wedge\left(\frac{\partial \xi_k}{\partial s}ds + \frac{\partial \xi_k}{\partial t}dt\right)$$

by virtue of the definition of $\xi(s,t) = g(s) - f(t)$

$$\Rightarrow \chi^*\omega = \frac{1}{\|g(s) - f(t)\|^3}\sum(-1)^{i+1}(g_i(s) - f_i(t))\left[\frac{\partial \xi_j}{\partial t}\frac{\partial \xi_k}{\partial s} - \frac{\partial \xi_k}{\partial t}\frac{\partial \xi_j}{\partial s}\right] \times ds \wedge dt \tag{114}$$

where $\partial\xi_j/\partial t = f_j'(t)$ and $\partial\xi_j/\partial s = g_j'(s)$.

Discussion of relationship (109). Let $f = c_1{}^1$ and $g = c_1{}^2$ be the topological components of two interacting fields. By virtue of Biot–Savart's law the work done by $\vec{H}$, i.e. the circulation of $\vec{H}$ around g, accounts for the corresponding interaction intensity. Consequently if

$$g_* S^1 \cap M = \phi \Rightarrow l(f, g) = 0 \tag{115}$$

there is no interaction. More generally if

$$f(S^1) \qquad \text{and} \qquad g(S^1) \tag{116}$$

can be separated by a hyperplane, again $l(f, g) = 0$. Otherwise stated, if $\{f, g\}$ and $\{f_0, g_0\}$ are two pairings of paths that are homotopic to each other, where

$$f_0\begin{cases} z = 0 \\ (x-2)^2 + y^2 - 1 = 0 \end{cases} \qquad \text{and} \qquad g_0\begin{cases} z = 0 \\ (x+2)^2 + y^2 - 1 = 0 \end{cases}$$

then again $l(f, g) = 0$. Finally

$$\omega^1 = H_i\, dx^i = 0 \text{ implies also } l(f, g) = 0 \tag{117}$$

A more precise version of Proposition 10 can be obtained as follows: For the compact oriented 2-manifold with boundary $M = c_2 \subset \mathbb{R}^3$ and $(x', y', z') \notin M$, let

$$\Omega = \int_M \frac{(x-x')dy\wedge dz + (y-y')dx\wedge dz + (z-z')dx\wedge dy}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \tag{118}$$

and suppose (x', y', z') on the same side of M as the vector $w_p \in \mathbb{R}_p{}^3 - T_p$ in such a way that $w_p, (v_1)_p, (v_2)_p$ is positively oriented in $\mathbb{R}_p{}^3$ when $(v_1)_p, (v_2)_p$ is positively oriented in T_p. Then we have the

Proposition 11. Let $g : S^1 \to \mathbb{R}^3$ and suppose that whenever $g(t) = p \in M$ one has $dg/dt \notin T_p$. Let n^+ be the number of intersections $g(S^1) \cap M$ where dg/dt points in the same direction as the vector w_p and n^- the number of other intersections. Then

$$n = n^+ - n^- = -\frac{I}{4\pi} \int_{S^1} g^* \, d\Omega \tag{119}$$

A proof of this proposition can be found in Crowell & Fox (1963).

Remark 28. It can be shown that formula (109) of Proposition 10 equals relationship (119), that is

$$n = l(f, g), \qquad \text{where } \partial M = f_*(S^1) \tag{120}$$

The physical interpretation of relationship (119) is the following. Consider Maxwell's equation $\operatorname{rot} \vec{H} = (4\pi/c)\vec{i}$. Since the current density $\vec{i}$ is zero outside $f_* S^1$, $\operatorname{rot} \vec{H} = 0$ permits the expression of $\vec{H}$ as the gradient of a magnetic scalar potential, i.e. $\vec{H} = -\operatorname{grad} \psi$.

Relationship (118) represents the solid angle subtended by M at $\vec{x}$ (Fig. 6).

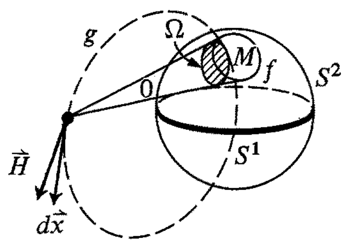

Figure 6.

According to electrodynamics:

$$\vec{H} = -\frac{I}{4\pi} \operatorname{grad} \Omega \Leftrightarrow \omega^1 = H_i \, dx^i = -\frac{I}{4\pi} d\Omega \tag{121}$$

or, which amounts to the same

$$H = \left(H_1 = -\frac{I}{4\pi} \frac{\partial \Omega}{\partial x} = \int_{S^1} f^* \left(\frac{(y - y')dz - (z - z')dy}{|(x, y, z)|^3} \right), \ldots, \right.$$

$$\left. H_3 = -\frac{I}{4\pi} \frac{\partial \Omega}{\partial z} = \int_{S^1} f^* \left(\frac{(x - x')dy - (y - y')dx}{|(x, y, z)|^3} \right) \right) \tag{122}$$

Remark 29. The multi-valuedness of the potential ψ is expressed in terms of

$$\psi_1 - \psi_2 = \oint \vec{H} \cdot d\vec{x} = \frac{4\pi}{c} I = \text{const.} \sum_k n_k I_k = n^+ - n^- \tag{123}$$

(for the unit currents I_k), which means that the potential ψ is single-valued if and only if f and g are not interlinked, i.e.

$$\psi_1 - \psi_2 = \oint \vec{H} \cdot d\vec{x} = 0 \tag{124}$$

The problem which arises now is, whether or not a gauge invariant theory, whose field variables are derived from the geometric structure of a principal toral bundle $P(M^4, SO(2))$, can account in a consistent way for concepts such as 'bare charge' or 'bare mass', 'renormalised charge (mass)' etc. These issues will be studied in terms of the following

---

Proposition 12. Let $\xi = P(M^4, SO(2))$ be a principal toral bundle whose connection is defined by the 1-form ω^1. Suppose

$$\Delta f \colon [0, 1] \subset \mathbb{R} \to \pi^{-1}(x_0) = F_{x_0} \subset P \tag{125}$$

be a loop (cf. Remark 30 below) along the circle $S^1 = SO(2)$, then

$$S^2 \in H_2(M^4)(\exists m \in \mathbb{Z}) \colon \frac{1}{4\pi} \int_{S^2} \omega^2 = m = w_a(\Delta f) \tag{126}$$

constitutes the charge enclosed in Δf. $w_a(\Delta f)$ is the winding number of Δf about a and

$$[\omega^2 = \sum F_{\mu\nu}\, dx^\mu\, dx^\nu] \in H^2(M^4, \mathbb{Z}) \tag{127}$$

is the first characteristic cohomology class $c_1(\xi)$ of $P(M^4, SO(2))$.

Remark 30. The mapping (125) Δf of I into the fibre F_{x_0} over $x_0 \in M^4$ is defined by

$$\Delta f(s) = \bar{f}(s, 1) \tag{128}$$

where

$$\bar{f} \in C^\infty(I^2, P(M^4, SO(2)) \tag{129}$$

denotes the lift of f to $P(M^4, SO(2))$, i.e. $\bar{f} \colon I \times I \to P$; $(s, t) \to \bar{f}(s, t)$ is given in terms of the properties (129), $\pi(\bar{f}(s, t)) = f(s, t) \in M^4$, and

$$\bar{f}(s, 0) = \bar{f}(0, t) = \bar{f}(1, t) = p_0 \in P, \qquad \pi(p_0) = x_0 \tag{130}$$

Δ constitutes the boundary homomorphism of the homotopy exact sequence

$$\to \Pi_2(P) \to \Pi_2(M^4) \overset{\Delta}{\to} \Pi_1(SO(2)) \to \Pi_1(P) \to \ldots \tag{131}$$

Remark 31. An important feature of $P(M^4, SO(2))$ is that its first Chern class be independent of the choice of the connection ω^1. In fact, let ω^1 and $\omega^{1'}$ be two connections in $P(M^4, SO(2)$, $\theta \in F^1(M^4)$ be such that $\pi^*\theta = \omega^1 - \omega^{1'}$ and $\Omega^2 = \pi^*(\omega^2)$ be the curvature; ω^2 represents the characteristic class of the bundle, then

$$\pi^*\, d\theta = d(\pi^*\theta) = d\omega^1 - d\omega^{1'} = \pi^*\omega^2 - \pi^*\omega^{2'} \tag{132}$$

and therefore

$$\omega^2 = \omega^{2'} + d\theta \tag{133}$$

i.e. ω^2 and $\omega^{2'}$ are cohomologous.

As regards the first Chern class of the principal circle bundle $P(\tilde{M}^4, SO(2))$, we can re-express the statement of Remark 15 in a more elegant fashion ($\tilde{M}^4$ is related to M^4 by means of the map (140), Proposition 13).

Proposition 12'. If the first Chern class of the bundle $\xi = P(\tilde{M}^4, SO(2))$ vanishes, i.e. $c_1(\xi) = 0$, there exists a connection ω^1, such that the local symmetry group is trivial, i.e. $\phi_{x_0} = I$.

Proof of Proposition 12. Let the map $h : \Pi_2(M^4) \overset{\text{into}}{\longrightarrow} H_2(M^4, \mathbb{Z})$ be a homomorphism of the second homotopy group of M^4 into the second homology group and $f \in C^\infty(I^2, M^4)$ be a representative of a homotopy class $[\alpha] \in \Pi_2(M^4, x_0)$ then, by virtue of (129),

$$\int_{h(\alpha)} \omega^2 = \int_f \omega^2 = \int_{\pi(\bar{f})} \omega^2 = \int_{\bar{f}} \pi^*\omega^2 = \int_{\bar{f}} \Omega^2 = \int_{\bar{f}} d\omega^1 = \int_{\partial\bar{f}} \omega^1,$$

$$\text{since } \Omega^2 = d\omega + \tfrac{1}{2}[\omega, \omega] = d\omega \tag{134}$$

Moreover, by virtue of (131), we obtain $\Delta f \in [\Delta\alpha] \in \Pi_1(SO(2))$ and since $\Pi_1(SO(2)) = \mathbb{Z}$ every element of $\Pi_1(SO(2))$ is a multiple of the generator γ of $\Pi_1(SO(2))$ and hence $\Delta\alpha = m\gamma$, $m \in \mathbb{Z}$. From (130) and (134) we infer $\int_{\partial\bar{f}} \omega^1 = \int_{\Delta f} \omega^1 = m \,.\, 4\pi$ (cf. Kobayashi, 1956).

Discussion. An important point is whether or not one is dealing with discrete or continuously variable charge, which amounts to studying the following three sets of equations:

$d\omega^2 = 0$	(135)	or	$d\omega^2 = 0$	(135')	or	$d\omega^2 = 4\pi\gamma$	(135'')
$d^*\omega^2 = 0$	(136)		$d^*\omega^2 = 4\pi\gamma^*$	(136')		$d^*\omega^2 = 4\pi\gamma^*$	(136'')

where (135')-(136') stand for Maxwell's equations γ is the magnetic monopol current. By virtue of the proof of Proposition 12 we are led to analyse the following two cases:

(1) Equation $d\omega^2 = 0$ (div $\vec{H} = 0$) is associated with the construction of *magnetic* charge g by means of the first Chern class $c_1(\xi) \in H^2(M^4, \mathbb{Z})$. The pole strength g which is measured to be

$$\frac{1}{4\pi} \int_{S^2} \omega^2 = m = w_a(\Delta f) \tag{126}$$

(cf. the adjoining Fig. 7) displays, that $m \in \mathbb{Z}$ is both, the value of the Chern class $c_1(\xi)$ and the magnetic charge enclosed in Δf.

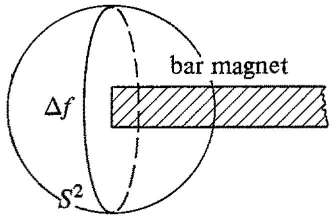

Figure 7.

(2) The occurrence of *discrete magnetic charge* is related to

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Proposition 13. Within an appropriate fibre bundle approach the following holds

$$w_a(\varphi) = \frac{1}{4\pi} \int_{\phi_* S^2} \tilde{\omega}^2 = w_a(\Delta f) = \frac{1}{4\pi} \int_{\varphi(S^2)} \omega \tag{137}$$

where

$$\phi : M^4 \to \tilde{M}^4, \qquad \phi \in C^\infty(M^4, \tilde{M}^4), \qquad S^2 \in H_2(M^4)$$
$$\varphi : S^2 \to S^2, \qquad \varphi \in C^\infty(S^2, \mathbb{R}^3) \tag{140}$$

and $w_a(\varphi)$ denotes the winding number of φ about $a = 0 \in \mathbb{R}^3$.

Proof. By

$$\phi_* : C_2(M^4) \to C_2(\tilde{M}^4) \tag{140}$$

one obtains

$$\int_{\phi_* S^2} \tilde{\omega}^2 = \int_{S^2} \phi^* \tilde{\omega}^2 = \int_{\varphi_*(S^2)} \omega^2 \tag{139}$$

where $\tilde{\omega}^2 \in F^2(\tilde{M}^4)$. On account of

$$w_a(\varphi) = \deg \mu_\varphi = w_0(\varphi) \tag{138}$$

where

$$\mu_{\varphi, 0}(x) \equiv \frac{\varphi(x) - 0}{\|\varphi(x) - 0\|} = \varphi(x)$$

$$\Rightarrow w_0(\varphi) = \deg \varphi = w_a(\varphi) \qquad \text{with} \qquad \deg \varphi = \frac{1}{4\pi} \int_{\varphi_* S^2} \omega^2 = w_a(\Delta f)$$

(Prop. 12) (141)

Remark 32. In Lemma 1 and the theorem of page 201 of my paper (v. Westenholz, 1971) the erroneous statement 'ϕ is a diffeomorphism' must be replaced by (140).

Remark 33. From Propositions 12 and 13 one infers that the dynamical coupling constants e or g admit the geometrical interpretation, that, being related to the characteristic class $c_1(\xi)$, they are obstructions to trivialising the principal bundle ξ.

Relationship (137) admits the following physical interpretation (Lubkin, 1971). Suppose the bar magnet of Fig. 7 becomes infinitely thin and infinitely

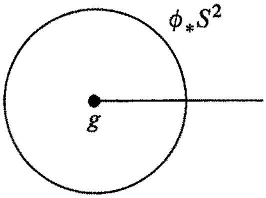

Figure 8.

permeable and centred at a. One thus defines a magnetic monopol field

$$H = \frac{g}{r^2} \tag{142}$$

and the half-line which represents the bar magnet may be regarded as a flux line.

Remark 34. By virtue of Dirac's condition,

$$\frac{eg}{\hbar c} = \tfrac{1}{2}n \tag{143}$$

the magnetic monopol charge (137) will be used in the sequel to define quantised electric charge.

The issue of magnetic monopoles is characterised by equation (135″) $d\omega^2 = 4\pi\gamma$, where γ is the conserved monopol current, i.e. $d\gamma = 0$. Within a gauge approach to electrodynamics one encounters the difficulty, however, that there exists no method for deriving Maxwell's equation (135″) from the geometric structure $P(M^4, SO(2))$, since $\pi^* d\omega^2 = d\Omega^2 = d(d\tilde{\omega}) = 0$ yields $d\omega^2 = 0$. This difficulty may be circumvented in the use of the bundle $P(M, SU(3))$ as the relevant structure for magnetic monopoles and by associating magnetic monopoles with the YM equations (73) $dB = 4\pi\omega^3$. A motivation for such an approach to magnetic monopoles is given by the papers of Schwinger (1968) and v. Westenholz (1970). In fact, Schwinger's baryon model, whose particles are magnetically charged quarks which carry fractional electric charge, is based upon the $SU(3)$-symmetry scheme. On the other hand, Loos (1967) has shown the existence of a solution to the YM equations (73) for a point charge within a gauge theory with non-Abelian gauge group. We therefore regard a monopol field as a non-Abelian gauge field, whose field equations are given by (73).

The formal symmetry between electric and magnetic charge expressed through equations (135″)–(136″) is violated by the great disparity between the charge units. Indeed, on account of Dirac's condition (143), $e^2/\hbar c = \alpha$ entails the least value allowed for g^2 to be $g^2/\hbar c = 137/4$. Thus, ultimately, the equation $d\omega^2 = 4\pi j$ only formally accounts for magnetic monopoles and should be abandoned in favour of the YM equation $dB = 4\pi\omega^3$. The great strength of magnetic attraction indicated by $g^2/\hbar c$ suggests that some (super) strong interaction is associated with the aforementioned approach. Moreover, the transition from a non-spherically symmetric and non-Abelian generalised Maxwell formalism to a framework consistent with a spherically symmetric monopol field $H = g/r^2$ might possibly account for a so-called breakdown mechanism of the unitary symmetry $SU(3)$. Mathematically this would amount to a reduction of the structure group $SU(3)$ to $SO(2)$,

$$f : P(M^4, SO(2)) \to P(M^4, SU(3)) \tag{144}$$

In order to recover the relevant geometrical objects for the description of the corresponding physical fields it suffices to consider the relationships $U_\alpha{}^\beta = A_\alpha{}^\beta + iB_\alpha{}^\beta$; $\omega^1 = \alpha^1 + i\beta^1$, $\Omega^2 = \alpha^2 + i\beta^2$ and equate all imaginary components equal to zero. The resulting objects correspond to $SO(2)$. These rather sketchy arguments will be developed more explicitly elsewhere.

Remark 35. The appropriate fibre bundle mentioned in Proposition 13 now turns out to be the principal bundle $P(M^4, SU(3))$.

The relationship which associates the dynamic charge coupling constant with the interaction intensity reduces to a formula between the linking number and the winding number according to

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Proposition 14. Let $f = f_* S^1$ and $g = g_* S^1$, $g, f \in \mathring{C}_1(\mathbb{R}^3)$ be two loops as specified in Proposition 10 (cf. Figs. 4 and 5), then

$$l(f, g) = k \, . \, w_a(\varphi) \qquad k \in \mathbb{Z} \tag{144}$$

Remark 36. Formula (144) makes physical sense, since $[k] = \mathrm{cm}^{-1}$, $[w_a(\varphi)] = $ charge and $[l(f, g)] = [\oint \vec{H} \, . \, d\vec{x}] = $ charge . cm^{-1}.

Proof of Proposition 14. We have to show the following relationships to hold:

$$w(\tilde{f}, a) = w_a(\varphi) \tag{145}$$

$$w(\tilde{f}, a) = w(f_c, a_c) \tag{146}$$

and

$$w(f_c, a_c) = l(f, g) \tag{147}$$

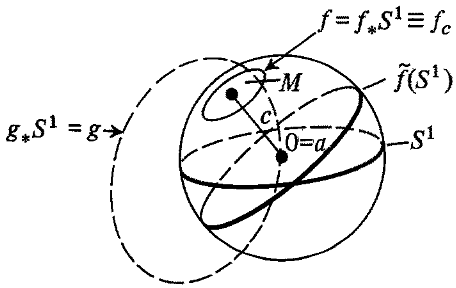

Figure 9.

That is, in the most general case:

$$l(f,g) = \sum_{P \in M \cap g(S^1)} w(P_i, \varphi) = \sum_{i=1}^{n} w(P_i, \varphi) = k \,.\, w_a(\varphi) \qquad (147')$$

where $M \cap g(S^1) = \{P_1, P_2 \ldots P_n; P_i \in M\}$.

(a) *Proof of* (145). Consider the mapping $f_c : S^1 \to f_c(S^1) = S_c \subset S^2$ where $f_0 = \tilde{f} : S^1 \to \tilde{f}(S^1)$ which 'rotates' S^1 (Fig. 9), $\deg f = m$. Let $\varphi : S^2 \to S^2$, $\varphi \in C^\infty(S^2, \mathbb{R}^3)$ be such that $(\varphi/S_c) \circ h_c = f_c$ where $S^1 \xrightarrow{h_c} S_c \xrightarrow{\varphi/S_c} S_c$ (cf. Fig. 9), with composite f_c.

The map h_c is defined by $\deg h_c = 1$ (in particular if $c = 0$, then $\varphi/S_0 = \tilde{f}$ and h_c = identity). We claim:

$$\deg \varphi = \deg \tilde{f} \qquad (149)$$

Indeed, let $c = 0$, then (149) is trivial, since $w(a, \tilde{f}) \overset{\text{def}}{=} \deg \mu$, where

$$\mu(x) = \frac{\tilde{f}(x) - a}{\|\tilde{f}(x) - a\|} = \frac{\tilde{f}(x)}{\|\tilde{f}(x)\|} = \tilde{f}(x) \Rightarrow \omega(a, \tilde{f}) = \deg \tilde{f} = \deg \varphi \qquad \text{since } \varphi/S_0 = \tilde{f}$$

$c \neq 0$ implies $\deg f_c = \deg(\varphi/S_c \circ h_c) = \deg \varphi/S_c \circ \deg h_c = \deg \varphi$.

(b) *Proof of* (146)

$w(\tilde{f}, a) = w(f_c, a_c) \Leftrightarrow \exists$ a homotopy $\{F_t, G_t\}$ such that $F_0 = \tilde{f}, G_0 = a$; $F_1 = f_c, G_1 = a_c$

Such a homotopy is given by

$$F_t(s) = tf_c(s) + (1-t)\tilde{f}(s) \qquad \forall s \in S^1$$

and

$$G_t(s) = ta_c + (1-t)a, \quad \text{since} \quad \left.\begin{matrix} a_c = a_c(s) \\ \text{and} \quad a = a(s) \end{matrix}\right\} \begin{matrix} \text{are constant functions} \\ G_t(s) \text{ is a constant function for } s \end{matrix}$$

(c) *Proof of formula* (148). By virtue of Proposition 11 and (119) we introduce the following sets:

$$A = \left\{ P \in M \cap g(S^1) \;\middle|\; \frac{dg}{dt} \text{ points in the same direction as } w_p \right\} = \{P_1, P_2, \ldots P_{n^+}\}$$

$$B = \left\{ P' \in M \cap g(S^1) \;\middle|\; \frac{dg}{dt} \text{ points in the opposite direction as } w_p \right\} = \{P'_1, P'_2 \ldots P'_{n^-}\}$$

That is, the integers n^+ and n^- are associated with A and B respectively. This leads us to distinguish between the following two cases: $\deg f_c = 1$ (f_c is a 1-1 function) and $\deg f_c = m \in \mathbb{Z}, m > 1$. Now, $\deg f_c = 1$ implies $w(P_c, f_c) = \deg \mu$ $=1$, where

$$\mu(s) = \frac{f(s) - a_c}{\|f(s) - a_c\|}$$

and

$$\left.\begin{array}{l}\sum_{P_i \in A} w(P_i, f_c) = \sum_{i=1}^{n^+} w(P_i, f_c) = n^+ \\ \sum_{P_i' \in B} w(P_i', f_c) = \sum_{i=1}^{n^-} w(P_i', f_c) = n^- \end{array}\right\} \Rightarrow (147)$$

$$l(f, g) = \sum_{i=1}^{n^+} w(P_i, f_c) - \sum_{j=1}^{n^-} w(P_j', f_c)$$

Formulae (145) and (146) yield

$$w(P, f_c) \equiv w(a_c, f_c) = w_a(\varphi) \tag{148}$$

If one substitutes (148) into (147) one infers $l(f, g) = (n^+ - n^-)w_a(\varphi) = kw_a(\varphi)$ which equals relationship (119), since $w_a(\varphi) = \deg f_c = 1$ by assumption. If -

$$\deg f_c = m \Rightarrow \deg f_c = \deg \mu = w(P, f_c) = m > 1 \qquad \forall P \in \mathring{M}$$

$\deg f_c = m$ means: $\forall P \in g(S^1) \cap M$, P stands for the set $P = \{P_1 \ldots P_m\}$ such that $w(P, f_c) = \Sigma_{i=1}^m \underbrace{w(P_i, f_c)}_{=1} = m$.

By definition of n^+ and n^- we get

$$n^+ = \sum_{\substack{P \in M \cap g(S) \\ P \in A}} \underbrace{w(P, f_c)}_{=m} = n_1 m, \qquad n^- = \sum_{\substack{P \in M \cap g(S) \\ P \in B}} w(P, f_c) = n_2 . m$$

therefore

$$l(f, g) = n^+ - n^- = n_1 m - n_2 . m = (n_1 - n_2) . m$$

and by virtue of (148) one obtains $l(f, g) = (n_1 - n_2)w_a(\varphi)$, $n_1 - n_2 \in \tilde{\mathbb{Z}}$, q.e.d.

Remark 37. Relationship (144) is a special case of $\Sigma_{g_i} l(f_c, g_i) = \tilde{k} . w_a(\varphi)$ where $\tilde{k} = \Sigma_i k_i$ (g_i denotes the number of field lines through a). Clearly $|c| = 1/\Sigma |k_i|$, $[\Sigma |k_i|] = \text{cm}^{-1}$, where $c \in [-1, 1]$. In fact, $|c| \to 0$ ($c \neq 0$) gives rise to an increasing number of field lines g_i which intersect with $M_c = c_2$ ($\partial M_c = f_c$) (refer to Fig. 9).

Discussion of Proposition 14. The relationship

$$\frac{1}{4\pi} \int_{S^2 = c_2} \tilde{\omega}^2 = w_a(\varphi) = 0 \tag{150}$$

implies that (a) no quantised charge is inside S^2 and (b) $l(f, g) = 0$, which obviously amounts to $L_I = ej_\mu A_\mu = 0$. In fact, the dynamic coupling constant e which is given by (137) or (141′) (cf. Remark 33) vanishes. Thus (150) is consistent with Remark 33 where charge was responsible for the obstruction to trivialising the principal bundle under consideration. Conversely, $l(f, g) = 0$

yields (a) $k = 0$ or (b) $w_a(\varphi) = 0$. Case (a) implies the cardinality of the sets A and B to be the same. Case (b), $w_a(\varphi) = 0$, amounts to

$$\operatorname{div} \vec{E} = 0 \tag{151}$$

which is just a special case of $d^*\omega^2 = 0$ (equation 136). Consequently, the *undressed* or *bare* charge e_0 which is involved in electromagnetic interaction processes will be identified with continuously variable (unquantised) charge

$$\int_{c^2} {}^*\omega^2 = 4\pi e_0 = \int_{c_2} {}^*E_{ij}\, dx^i \wedge dx^j$$
$$x^0 = \text{const.} \tag{152}$$

The charge (152) will be associated with a multiply connected topology trapping electric lines of force (v. Westenholz, 1971; Misner & Wheeler, 1957) in the following sense: If the interiors of two solid 2-spheres S^2 of equal radius in the $T =$ const. hyperplane are removed and the appropriate points on the surfaces S^2 are identified, one obtains the pattern of an electric dipole, i.e. $\operatorname{div} \vec{E} = 0$, provided the charge e and $-e$, respectively, are assigned to these spheres. Thus, charge appears as a non-local manifestation of source-free electrodynamics in a multiply-connected topological space. Such a non-local picture of charge is familiar from vacuum polarisation whose net effect is to spread out the effective charge over distances of the order $\hbar/mc$. This characterises non-locality.

Remark 38. The validity of equation $w_a(\varphi) = 0$, which accounts for the absence of *quantised* charge in S^2, by no means contradicts the existence of continuously variable charge (equation (152)). On the contrary, this type of charge, being a manifestation of source-free electrodynamics, is, by our foregoing construction, associated with an empty S^2.

Within the framework of YM and AB fields the interaction constant $\alpha = e^2/\hbar c$ can be estimated. This will be established now. We assume that

(a) units of particle physics (i.e. quarks) are represented by elementary loops (cf. Fig. 4);
(b) these elementary loops are supposed to be quantised in the following sense: let

$$A_\mu = (\varphi_{\mathrm{AB}}, \vec{A}_{\mathrm{AB}}) \tag{153}$$

denote the 4-potential representing an AB field, where

$$\vec{A}_{\mathrm{AB}} = -\frac{\hbar c}{e}\frac{\partial\vartheta}{\partial x^k} \tag{154}$$

and

$$\varphi_{\mathrm{AB}} = \frac{\hbar c}{e}\frac{\partial\vartheta}{\partial x^0} = \frac{\hbar}{e}\frac{\partial\vartheta}{\partial t} \tag{155}$$

$\vartheta(x^\mu)$ is a continuous function of space time. Then the following holds:

Proposition 15. The vector potential $\vec{A}_{\mathrm{AB}}$ produces the quantised flux

$$\phi_{\mathrm{AB}} = \frac{\hbar c}{e} \tag{156}$$

and the scalar potential (155) yields the quantity

$$\frac{\partial \vartheta}{\partial t} = k \frac{\Delta E}{\hbar} \tag{157}$$

where $k = e^2/(2mc^2 . r)$ is a dimensionless constant. ΔE denotes the energy uncertainty during the intermediate state of some virtual process which is characterised in terms of AB fields.

'*Proof by quotation*'. By virtue of the correspondence principle between AB fields and virtual processes, we denote by

$$\Delta E = \Delta_1 E + \Delta_2 E \tag{158}$$

the energy uncertainty associated with the fictitious intermediate state, where

$$\Delta_1 E \sim I \int_{c_1 = f} \omega^1; \qquad \omega^1 = \vec{A} . d\vec{x} \qquad \text{and} \qquad I = -\frac{e}{\tau} \tag{159}$$

denotes the steady current around the loop f.

$$k_1 . \Delta_2 E = e \Delta \varphi_{\mathrm{AB}}; \tag{160}$$

$\Delta\varphi_{\mathrm{AB}}$ is the uncertainty of the potential difference (Furry & Ramsey, 1960).

$\Delta_1 E$ is associated with the vector potential AB effect; $\Delta_2 E$ is the contribution of the uncertainty due to $\Delta\varphi_{\mathrm{AB}}$. Now, since

$$S = e \int \varphi_{\mathrm{AB}}(t) . dt = 2 \int T . dt \tag{161}$$

and

$$\Delta\varphi_{\mathrm{AB}} = k_2 \varphi_{\mathrm{AB}}; \qquad 2T \sim \Delta_2 E \tag{162}$$

we have (160). Moreover,

$$\frac{\hbar}{e} \frac{\partial \vartheta}{\partial t} = \frac{\hbar}{e} \left(k_3 \frac{\Delta_2 E}{\hbar} \right) \tag{163}$$

k_3 unknown. The uncertainty in the phase difference can be shown to be

$$\Delta\vartheta = \frac{e \Delta t . \Delta\varphi_{\mathrm{AB}}}{\hbar} \tag{164}$$

where Δt denotes an infinitely short lifetime. By (160) and (164) we obtain $(\Delta\vartheta/\Delta t) = k_1(\Delta_2 E/\hbar)$, by (155) and (163), $\Delta\varphi_{\mathrm{AB}} = k_2 k_3 (\Delta_2 E/e) \Rightarrow k_1 = k_2 . k_3$.

Now, for a fixed potential φ_{AB}, the evaluation of $\Delta\varphi_{AB}$ and hence of k_2 is possible. Therefore it remains to either determine k_1 or k_3. If φ_{AB} is brought to the fixed potential $\varphi_{AB} = (e/r)|_{S_2} = e$ with respect to the AB-scalar potential effect, then $k_3 = \phi/\phi_{AB}$ and conversely.

$$\phi = 2\pi \int_R^\infty Hr\,dr = \frac{e\hbar}{2mc} \cdot \frac{2\pi}{r}$$

denotes the flux

$$\vec{H} = \frac{3\vec{\mu}.\vec{r}}{\vec{r}^5}\vec{r} - \frac{\vec{\mu}}{r^3} = \frac{I}{4\pi}\int \frac{d\vec{y} \times (\vec{y} - \vec{x})}{\|\vec{y} - \vec{x}\|} \quad \text{(proposition 10)} \qquad (110)$$

and

$$\phi_{AB} = \frac{\hbar c}{e}$$

by

$$\varphi_{AB}(t) \cdot \frac{e}{\hbar} \cdot \frac{\hbar}{\Delta_2 E} = k_3 \qquad (163)$$

That is, in considering virtual states which violate the energy principle by an amount of (164), $\Delta E \sim 2m_0c^2$, one obtains

$$k_3 = \frac{e^2}{2 . m_0 c^2 . r} = \frac{\phi}{\phi_{AB}} = \left.\frac{e^2}{2m_0c^2}\right|_{S^2} \qquad (165)$$

In terms of formulae (155) and (163) this yields the correct value of the fine structure constant, since

$$\frac{\partial\vartheta}{\partial t} = \frac{e^2}{2m_0c^2} \cdot \frac{2m_0c^2}{\hbar} \Rightarrow \boxed{\frac{1}{c}\frac{\partial\vartheta}{\partial t} = \frac{e}{\hbar c}\varphi_{AB} = \left.\frac{e^2}{\hbar c}\right|_{S^2}} \qquad (166)$$

Remark 39. In the approach to elementary particle physics in terms of quantised flux loops the reverse situation to Dirac's electron theory applies in that the magnetic moment $\mu = eh/2mc$ is assigned to these loops (cf. Jehle, 1971).

Remark 40. An *a posteriori* justification for a description of interaction in terms of AB fields and magnetic charge is given by $\Delta\vartheta = (e . g/r^2)r^2 . A$ (cf. Lubkin, 1971) which represents the AB phase shift around the loop f (r^2A denotes the area enclosed by the loop f). It is 0 for $g = 0$ which corresponds to the absence of interaction.

Discussion of Proposition 15. During a virtual state corresponding to the interaction $H_I = e\bar{\psi}\gamma^\mu\psi A_\mu$, the AB potential A^μ_{AB} gives rise to the electromagnetic interaction constant $(e^2/\hbar . c)|_{S^2}$ provided one applies the potential $\varphi = (e/r)|_{S^2} = e$ to the AB-device which corresponds to the scalar effect. The value of the potential equals the potential of points on S^2 with respect to an AB source located at the origin. Now, consider the AB fields (25),

$\omega^2 = E_i\,dx^i\,dx^0 + {*H_{ij}}\,dx^i\,dx^j$, and (26), $*\omega^2 = H_i\,dx^i\,dx^0 + {*E_{ij}}\,dx^i\,dx^j$, where $d\omega^2 = d{*\omega^2} = 0$ and let

(a) $x^0 = \text{const.}$ The contributions of (25) and (26) are given by

$$\phi^0_{AB} = \int_{c^2} {*H_{ij}}\,dx^i\,dx^j \tag{167}$$

and

$$4\pi e_0 = \int_{c^2} {*E_{ij}}\,dx^i\,dx^j \tag{152}$$

the AB-flux and undressed charge, respectively.

(b) $dx^0 \neq 0$ implies

$$E_i\,dx^i\,dx^0 \neq 0 \tag{168}$$

and

$$H_i\,dx^i\,dx^0 \neq 0 \tag{169}$$

This entails that the interaction process, which is defined by the quantities

$$l(f, g) = \int_g H_i\,.\,dx^i \tag{109}$$

and

$$mc^2 = e\int E_i\,.\,dx^i \tag{170}$$

is described, during an infinitely short lifetime dx^0, by the AB fields (25) and (26). That is, the magnetic field $\vec{H} = -(I/4\pi)\,\text{grad}\,\Omega$ mediates the interaction between some quark represented by $f = f_* S^1$ and the photon field, whose contribution to the interaction is given by $\Delta_1 E$. The electric field determines the energy uncertainty $\Delta_2 E$ by (170). As regards the total energy violation during the fictitious intermediate state, the following holds: The interaction energy $H_I = ej^\mu\,.\,A_\mu$ can be associated with eight possible virtual processes whose constituents are e^+, e^- and photons:

$e^- \to e^- + \gamma$	$e^- + \gamma \to e^-$	$e^+ \to e^+ + \gamma$	$e^+ + \gamma \to e^+$
$\gamma \to e^+ + e^-$	$0 \to \gamma + e^+ + e^-$	$e^+ + e^- \to \gamma$	$\gamma + e^+ + e^- \to 0$

These eight processes are represented by essentially one *single* type of Feynman graph. Proposition 15 refers to those of these processes which do not satisfy energy conservation, i.e. $\gamma + e^+ + e^- \to 0$ and $0 \to \gamma + e^+ + e^-$. In both cases $\Delta E \sim 2m_0c^2$ provided we assume the emission or absorption of *very soft photons*.

Remark 41. There is no Lorentz force acting during the intermediate state, since $e\vec{E} + (e/c)[\vec{v}, \vec{H}] = 0$ for AB fields.

Remark 42. There is a 'source lepton' located at 0, whose magnetic dipole field lines correspond to loops $g = g_* S^1$ (i.e. these field lines represent quantised flux lines). The field lepton e^- produces a steady current through the loop $f = f_* S^1$ which gives rise to the field $H_i \, . \, dx^i$. Thus, some field lines due to the AB effect will coincide with field lines due to the current through f.

Finally we wish to study some typical virtual process, say $e^- + \gamma \to e^-$. We proceed as follows:

(I) $w_a(\varphi) = 0$, i.e. $l(f, g) = 0$ holds all the time. There is no interaction at all, which corresponds to the transition $e^- \to e^-$, i.e. the trivial reaction whose scattering operator is $S = I$.

(II) The interaction process decomposes into

(1) *The initial state.* The scattering of the two particles e^- and γ is represented by the Feynman graph (Fig. 10). During the initial state, the non-local bare charge $e_0 = e_i$ is associated with the source-free Maxwell field $(^*\omega^2, c_2)$ in terms of

$$(^*\omega^2, c_2) \to \int_c {}^*\omega^2 = 4\pi e_0$$

$$H^2(M^4) \times H_2(M^4) \to \mathbb{R} \tag{171}$$

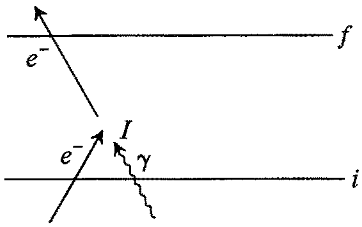

Figure 10.

(2) *The intermediate state.* As a result of the interaction $H_I = ej^\mu A_\mu$ or equivalently $l(f, g) \neq 0$, $w_a(\varphi) \neq 0$, the following holds. By virtue of its interaction with the radiation field, the particle e^- has acquired the electromagnetic mass δm as a consequence of the field H, which exists during $dx^0 \neq 0$ (Fig. 6), $mv^2/2 = \frac{1}{2} \int H^2 \, d^3x \Rightarrow \delta m_1$ (v is the velocity of the electron $-e$ around f during the virtual state (cf. Remark 45). The contribution δm_2 stems from $\vec{E}$, therefore $H_{\text{rad}} = \frac{1}{2} \int (E^2 + H^2) \, d^3x$ and $e_{\text{exp}} = e_0 + \delta e$, $m_{\text{exp}} = m_0 + \delta m$.

(3) *The final state.* The 'dressed' charge which the particle has acquired during the intermediate state is described by virtue of

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Proposition 16. The dressed charge $e_f = e_0 + \delta e$ which corresponds to the virtual process $e^- + \gamma \to e^-$ is given by

$$4\pi e_f = \int {}^*\omega^2 = \int dx^1 \wedge dx^2 = \text{const.} \tag{173}$$

Proof. The radiation field corresponding to the intermediate state is subject to the condition

$$\exists X : i_X\omega^2 = i_X{}^*\omega^2 = 0 \tag{174}$$

where $X = (X_0, (c/4\pi)(\vec{E} \wedge \vec{H})) = (X_0, \vec{S})$ ($\vec{S}$: Pointing vector).

$$\begin{matrix}\text{Rank } \omega^2 < 4 \\ \text{Rank } {}^*\omega^2 < 4\end{matrix} \quad \text{i.e.} \det(F_{\mu\nu}) = \det({}^*F_{\mu\nu}) = 0$$

and the skew-symmetry of $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ implies the rank of F to be even, i.e. Rank $\omega^2 = 2$ or 0. Since neither ω^2 nor ${}^*\omega^2$ are non-vanishing fields, they must be monomials of the form

$$^*\omega^2 = dx^i \wedge dx^j \tag{175}$$

Clearly

$$d\omega^2 = d^*\omega^2 = 0 \tag{176}$$

Since ${}^*\omega^2 = {}^*E_{ij}\, dx^i\, dx^j$, relationship (175) represents a uniform electric field in the x-direction, i.e. ${}^*E_{12} = E_3 = E_2 = 0$, ${}^*E_{23} = E_1 = \text{const}$.

Discussion of Proposition 16. By equation (173) it turns out that the dressed charge as defined by relationship (171) is again non-local in character. This corresponds physically to the polarisation phenomenon where the charge e_0, as a result of the interaction, surrounds itself by a cloud of charged particles. Some of these escape to infinity leaving a net charge of $-\delta e$ in the part of the cloud spread out over a distance of $\hbar/mc$.

Remark 43. By virtue of the foregoing reasonings, the radiation field involved in the interaction must take the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & -\eta E \\ 0 & 0 & 0 & 0 \\ 0 & \eta E & 0 & 0 \end{pmatrix} \qquad {}^*F_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\eta E & 0 \\ 0 & 0 & 0 & \eta E \\ -\eta E & 0 & 0 & 0 \\ 0 & \eta E & 0 & 0 \end{pmatrix} \qquad |\eta| = 1$$

Remark 44. Equation (174) states that the radiation field is associated with the direction defined by the Poynting vector, i.e.

$$i(S)\omega = F_{\mu\nu}(X^\mu\, dx^\nu - X^\nu\, dx^\mu) = 0 \qquad X^\mu = (S^0, \vec{S}) S^0 = |\vec{E}| = |\vec{H}| \Rightarrow F_{\mu\nu}X^\nu = 0$$

That is, the flux of $F_{\mu\nu}$ through a plan which contains $\vec{S}$ vanishes.

Remark 45. The quantity δm is assumed to be of electromagnetic origin and can be obtained formally as follows. Set

$$T = \frac{m}{2} v^2 \cong \frac{\delta m}{2} v^2 \qquad \text{where} \qquad \frac{\delta m}{2} v^2 = 1/2 \int H^2 \,.\, d^3x \tag{177}$$

$$\therefore = \text{const} \int_1^\infty \frac{dr}{r^2} \int_0^\pi \sin^3 \vartheta\, d\vartheta \int_0^{2\pi} d\varphi \Rightarrow \delta m < \infty \tag{178}$$

The magnetic field

$$|\vec{H}| = \frac{ev}{4\pi} \cdot \frac{\sin \vartheta}{r^2} \tag{179}$$

satisfies Biot–Savart's law and acts during the short lifetime $dx^0 \neq 0$. The quantities (177) and (178) correspond to the uniform motion of the electron $-e$ in f which produces the field (179).

Conclusion

It is known that in electron theory one distinguishes between the charge of the undressed and the experimental electron. The factors of conversion are logarithmically divergent. Because of this it is claimed that only the renormalised theory has any physical significance. However, within our continuum picture it turns out that there will be no infinite factor of conversion and thus no renormalisation will be needed.

References

Aharonow, Y. and Bohm, D. (1959). *Physical Review*, **115**, 485.
Ambrose, W. and Singer, I. M. (1953). *Transactions of the American Mathematical Society*, **75**, 429.
Crowell, R. and Fox, R. (1963). *Introduction to Knot Theory*. Mc. Ginn, N.Y.
Furry, W. and Ramsey, N. (1960). *Physical Review*, **118** (3), 623.
Gallissot, F. (1951). *Annales de l'Institut Fourier*, **3**, 145.
Goldschmidt, H. and Sternberg, S. (1973). *Annales de l'Institut Fourier*, **23** (1), 203.
Hermann, R. (1968). *Differential Geometry and the Calculus of Variations*. Academic Press, New York.
Hermann, R. (1970). *Vector Bundles in Mathematical Physics*. Benjamin.
Hiley, B. and Stuart, A. (1971). *International Journal of Theoretical Physics*, Vol. 4, No. 4, p. 247.
Jehle, H. (1971). *Physical Review*, **3** (2), 306.
Kobayashi, S. (1956). *Tohoku Mathematical Journal*, **8**, 29.
Kobayashi, S. and Nomizu, K. (1963). *Foundations of Differential Geometry I*. Interscience.
Lichnerowicz, A. (1964). *Bulletin de la Société mathématique de France*, **92**, 11.
Loos, H. G. (1967). *Journal of Mathematical Physics*, **8** (10), 2114.
Lubkin, E. (1971). *American Journal of Physics*, **39**, 94.
Mandelstam, S. (1962). *Annals of Physics*, **19**, 1.
Misner and Wheeler, A. (1957). *Annals of Physics (N.Y.)*, **2**, 525.
Schwinger, J. (1968). *Physical Review*, **173**, 1536.
Trautman, A. (1968). Preprint No. 47, *Fibre Bundles in Physics*. Torun.
Utiyama, R. (1956). *Physical Review*, **101** (4), 1597.
Voros, 15^e Rencontre entre Physiciens Theoriciens et Mathematiciens Université de Strasbourg, 1972. Preprint.
Westenholz, C. v. (1970). *Annalen der Physik*, **25** (4), 337.
Westenholz, C. v. (1971). *Annales de l'Institut Henri Poincaré*, **XV** (3), 189.
Westenholz, C. v. (1972). *Annales de l'Institut Henri Poincaré*, **XVII** (2), 159.
Westenholz, C. v. (1973). Lecture Notes, University of Frankfurt.
Westenholz, C. v. (1973). On the significance of electromagnetic potentials in the quantum theory. *Annales de l'Institut Henri Poincaré*, **XVIII** (4), 353.